MÉTODOS NUMÉRICOS (de TAYLOR)

$$y' = f(t,y)$$

$$y'' = f_t + f_y y' = f_t + f_y \cdot f$$

$$y''' = f_{tt} + f_{ty} f + (f_{yt} + f_{yy} f) f + f_y (f_t + f_y f) =$$

$$= f_{tt} + 2 f_{ty} f + f_{yy} f^2 + f_y \cdot f_t + f_y^2 f.$$

...

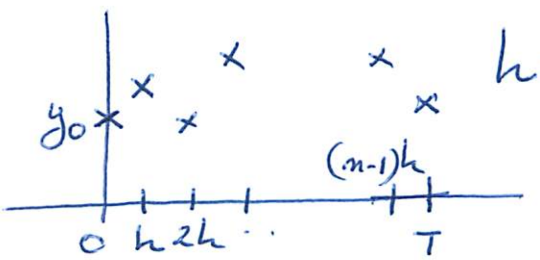
$$y(t+h) = y(t) + h\, y'(t) + \frac{h^2}{2!} y''(t) + \frac{h^3}{3!} y'''(t) + \cdots$$

Método de Taylor de 1^er^ orden (Euler)

$$y(t+h) \simeq y(t) + h\, y'(t) = y(t) + h f(t, y(t))$$

Esquema numérico $\left\{ \begin{array}{l} y' = f(t,y) \\ y(0) = y_0 \end{array} \right.$

$$(E) \left\{ \begin{array}{l} w_{i+1} = w_i + h f(t_i, w_i) \\ w_0 = y_0 \end{array} \right.$$



$$h = \frac{T}{n}$$

$$\downarrow$$

$$\left(\begin{array}{l} w_0 = y(0) = y_0 \\ w_1 \simeq y(t_1) = y(h) \\ \vdots \\ w_n = y(t_n) = y(nh) = y(T) \end{array} \right)$$

$$\boxed{w(n) \simeq y(T)}$$

$$\boxed{t_i = i \cdot h}$$

Método de Taylor de 2º orden

$$y(t+h) \simeq y(t) + h\,y'(t) + \frac{h^2}{2!}y''(t) = y(t) + h\,f(t,y) + \frac{h^2}{2}(f_t + f_y f)$$

$$(T2)\begin{cases} w_{i+1} = w_i + h\,f + \frac{h^2}{2}(f_t + f_y \cdot f) \\ w_0 = y_0 \end{cases}$$

donde f, f_t, f_y están calculados en (t_i, w_i).

Programa

Entrada : T, n, y_0, f, f_t, f_y

Salida : $(w_0, w_1, \ldots, w_n)$ (Aprox. a la solución)

Método de Taylor de orden 3

$$y(t+h) \simeq y(t) + h\,f + \frac{h^2}{2}(f_t + f_y f) + \frac{h^3}{6}(f_{tt} + 2f_{ty} f + f_{yy} f^2 + f_y f_t + f_y^2 f)$$

$$(T3)\begin{cases} w_{i+1} = w_i + h\,f + \frac{h^2}{2}(f_t + f_y f) + \frac{h^3}{6}(f_{tt} + 2f_{ty} f + f_{yy} f^2 + f_y f_t + f_y^2 f) \\ w_0 = y_0 \end{cases}$$

donde $f, f_t, f_y, f_{tt}, f_{ty}, f_{yy}$ están calculadas en (t_i, w_i)

Programa : Entrada $\rightarrow T, n, y_0, f, f_t, f_y, f_{tt}, f_{ty}, f_{yy}$

Salida: $(w_0, w_1, \ldots, w_n)$

Análisis del método de EULER (Taylor de orden 1)

Sea $y(t)$ la única solución al PVI bien planteado

$$\begin{cases} y' = f(t,y) \\ y(a) = \alpha \end{cases} \qquad a \leq t \leq b$$

y sean $w_0, w_1, \dots, w_N$ las aprox. generadas por el método de Euler.

TEOREMA Si f satisface una condición de Lipschitz con constante $L>0$ en $D = \{(t,y) \,/\, a \leq t \leq b,\ -\infty < y < \infty\}$, ie,

$$|f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2| \qquad \forall (t,y_1),(t,y_2) \in D$$

y existe una constante $M>0$ tal que:

$$|y''(t)| \leq M \quad , \ \forall t \in [a,b]$$

entonces

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{(t_i - a)} - 1\right] \qquad i = 1, \dots, N$$

donde $h = \dfrac{b-a}{N}$, $t_i = a + ih$.

Esquema que incluye los errores de redondeo en el método de Euler.

$$(EA)\begin{cases} \hat{w}_0 = \hat{\alpha} & (\text{aprox. del orden. a } \alpha) \\ \hat{w}_{i+1} = \hat{w}_i + h\, f(t_i, \hat{w}_i) + \delta_{i+1} \end{cases}$$

(δ_i controla el error cometido en la computación)

<u>TEOREMA</u> Sea $y(t)$ la solución única del PVI bien planteado $\begin{cases} y' = f(t,y) \\ y(a) = \alpha \end{cases}$ $a \leq t \leq b$

y $\hat{w}_0, \hat{w}_1, \dots, \hat{w}_N$ las aproximaciones obtenidas usando (EA). Si $|\delta_i| < \delta$ para $i = 0, 1, \dots, N$, donde $\delta_0 = \hat{\alpha} - \alpha$ y si se satisfacen las hipótesis del teorema anterior, se tiene que

$$|y(t_i) - \hat{w}_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i - a)} - 1\right] + $$
$$+ |\delta_0| e^{L(t_i - a)} \qquad i = 0, 1, \dots, N$$

Ejemplo 1: Dibujar la solución del (PVI) $\left\{ \begin{array}{l} y' = \sin(x^2) \\ y(0) = 1 \end{array} \right.$,

para $0 \leq x \leq 1$

Integrando la ec. dif. queda.

$y(x) = \int_0^x \sin(t^2)dt + C$, usando condición inicial

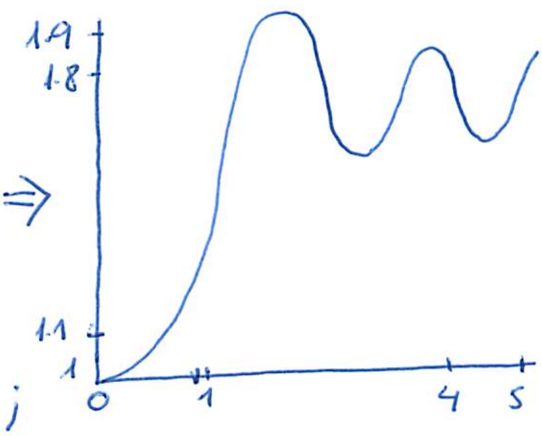
$y(0) = 1$, $C = 1$. Por tanto

$$y(x) = \int_0^1 \sin(t^2)dt + 1$$

Comandos MATLAB

```
>> f = inline('sin(x.^2)');
>> x=0:.01:5;
>> size(x)
   → 1 501
>> for i=1:501
     y(i)=1+quad(f,0,x(i));
   end
>> plot(x,y)
```

$\Rightarrow$



Si se hubiera usado un fichero .m para definir la función

f.m → $\boxed{\begin{array}{l}\text{function } y=f(x) \\ y=\sin(x.\hat{}2);\end{array}}$

El comando de integración usado "quad" tendría la siguiente sintaxis:

`>> quad('f',0,x(i))` o `>> quad(@f,0,x(i))`

Ejemplo 2: En el modelo de población tipo logístico (Verhulst) para la población de Estados Unidos hecho en 1920, se usó el (PVI)

$$\begin{cases} P'(t) = r P(t) \left(1 - P(t)/k\right) \\ P(0) = P_0 \end{cases}$$

usando la estimación $r = 0.0318$ (crecimiento medio), $K = 200$ millones (capacidad soporte). Es conocido que $P(0) = 3.9$ millones (donde hemos identificado $t=0$ año con el año 1790).

a) Usar método de Euler con tamaño de paso $h = 0.1$ para estimar la población de E.U. en los años 1850 ($t=6$), 1900 ($t=110$) y 1990 ($t=200$).

b) Repetir ~~el~~ apartado a) con paso $h = 0.01$.

c) La solución exacta del (PVI) es $P(t) = \dfrac{K}{1 + \left(\frac{K}{P_0} - 1\right) e^{-rt}}$

En el mismo plano dibujar la solución exacta $P(t)$, junto con las dos aprox. obtenidas en a) y b) para $0 \le t \le 200$.

```
>> f=inline('0.0318*P*(1-P/200)');
```

a)
```
>> t=0:0.1:200;
>> size(t)
→ 1  2001
>> P(1)=3.9;
>> for n=1:2000
     P(n+1)=P(n)+0.1*f(P(n));
   end
```
$\}$ Euler

```
>> P(601), P(1101), P(2001)
→ 23.5827, 79.1281, 183.9685
```

b) >> tb= 0 : 0.01 : 200; size(t)

→ 1   20001

>> Pb(1) = 3.9;

>> for n = 1 : 20000

Pb(n+1) = Pb(n) + 0.01 * f(Pb(n));

end

>> Pb(6001), Pb(1101), Pb(20001).

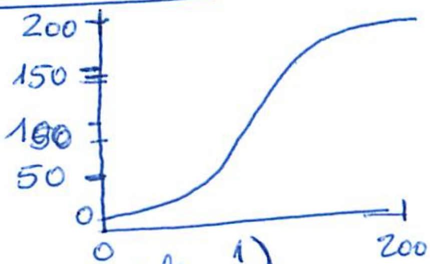
→ 23.6331 , 79.3010 , 183.9969

c) Almacenamos la solución exacta en fichero. sol.m

```
function y = sol(t)
y = 200./(1+(200/3.9 - 1).*exp(-0.0318*t));
```

>> plot (t, P), hold on, plot (tb, Pb)

>> plot (tb, sol(tb))   ⇒



>> xlabel ('Años después de 1790')

>> ylabel ('Estimación de la pobl. de E.U. en millones')

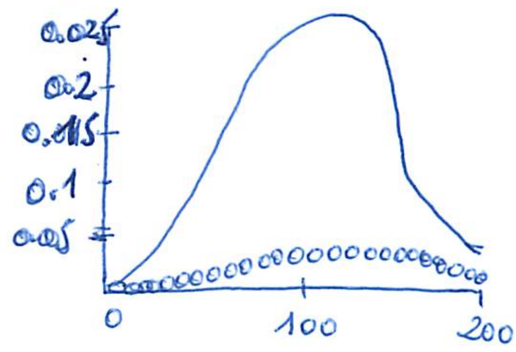
Puesto que los tres gráficos son indistinguibles, damos un gráfico de los errores cometidos en las aprox. a) y b).

>> hold off,

>> plot (t, abs(sol(t)-P), tb, abs(sol(tb)-Pb), 'o')

>> xlabel ('Años después de 1790')

>> ylabel ('Millones')   ⇒



>> title ('Gráfico de los errores')

Programa 1: Escribir un fichero "euler.m" para resolver por el método de Euler el (P.V.I) $\begin{cases} y' = f(t,y) \\ y(a) = y_0 \end{cases}$

euler.m

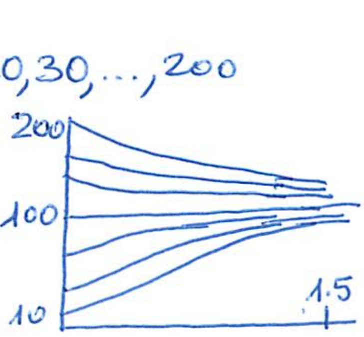
```
function [t,y]=euler(f,a,b,yo,h)
  % variables de entrada f, a,b,yo,h ;
  % f (función de la E.Dif), a (tiempo de inicio)
  % b (tiempo final); yo (valor de y(a));
  % h (~~tamaño~~ número del pasos).
  % Variables de salidad t (vector tiempo)
  % y (vector solución en los tiempos t).
  % f es una función de 2-variables (f(t,x))
  % definida en un fichero f.m (o mediante "inline")

  t(1)=a; y(1)=yo;
  h = (b-a)/n  % tamaño del paso.

  for i=2:n
      t(i)=t(i-1)+h;
      y(i)=y(i-1)+h*feval(f,t(i-1),y(i-1));
  end
```

Ejemplo 3: Usando programa 1, dibujar varias aprox. de Euler de $\begin{cases} P' = 2.2\,P(1-P/100) \\ P(0) = P_0 \qquad 0 \le t \le 1.5 \end{cases}$ con $P_0 = 10, 20, 30, \dots, 200$

```
>> f=inline('2.2*P*(1-P/100)','t','P')
>> hold on
    >> for i=10:10:200
       [t,yi]=euler(f,0,1.5,i,15);
       plot(t,yi)
       end
```



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MN Programa 2: Escribir un fichero «rungekutta.m» para calcular por el método de Runge-Kutta clásico de orden 4, la solución del (P.V.I) $\begin{cases} y' = f(t,y) \\ y(a) = y_0 \end{cases}$

rungekutta.m

```
function [t,y] = rungekutta(f,a,b,yo,n)
% las misma entradas y salidas de «euler.m»
  t(1)=a ; y(1)=y0;
  h = (b-a)/n;
  for i=2:n
      t(i) = t(i-1)+h;
      k1 = feval (f,t(i-1),y(i-1));
      k2 = feval (f, t(i-1)+0.5*h , y(i-1)+0.5*h*k1);
      k3 = feval (f,t(i-1)+0.5*h , y(i-1)+0.5*h*k2);
      k4 = feval (f,t(i-1)+h , y(i-1)+h*k3);
      y(i) = y(i-1) + 1/6*h*(k1+2*k2+2*k3+k4);
  end
```

Ejercicio 4: Desde que un paracaidista se lanza desde una avión hasta que se abre el paracaídas, la resistencia del aire es proporcional a $|v(t)|^{1.5}$, y la velocidad máxima que alcanza es de 80 mph.

a) Hacer un gráfico de la velocidad de caída durante los 10 primeros segundos (usando Runge-Kutta con tamaño de paso $h=0.01$ seg. ($n=1000$)). En el mismo dibujo incluir la velocidad de caída si no hay resistencia del aire.

b) ¿Cuántos segundos (con aprox. de ± 0.01 seg) tarda el paracaidista en alcanzar 60 mph?

Ley de Newton $\quad m\ddot{x} = -mg + k|\dot{x}|^{1.5}$

$\dot{v} = -g + c|v|^{1.5}$ (ec. para la velocidad, $v = v(t)$)

$g = 32.1740 \ ft/seg^2$, $1 \ m = 5280 \ ft$

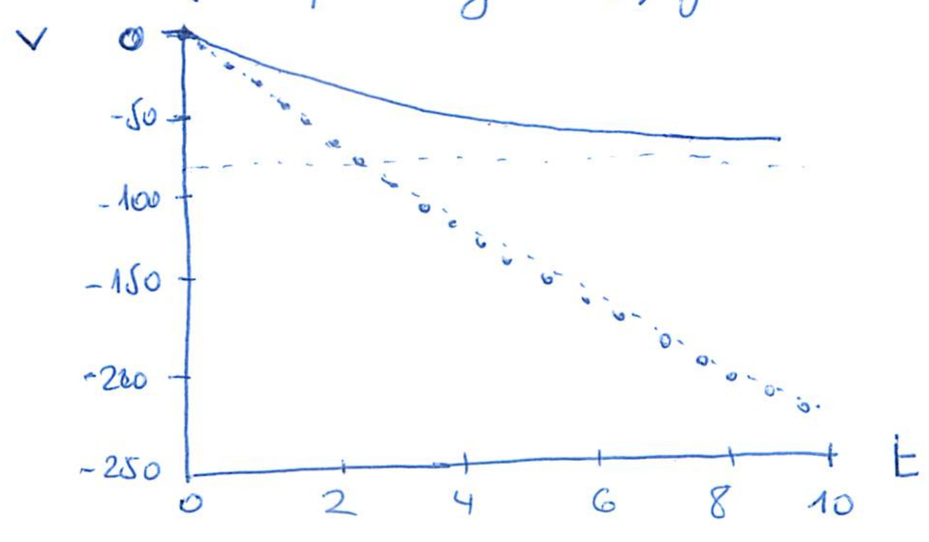
$\dot{v}(t) = 0$ (velocidad máx) $v(t) = 80 \ m/h$

$$v(t) = 80\left(\frac{m}{h}\right)\left(\frac{5280 \ ft}{1 \ m}\right)\left(\frac{1 \ h}{3600 \ seg}\right) \simeq \frac{352}{3} \ ft/seg$$

Por tanto $\quad c \simeq 32.1740/(352/3)^{1.5}$

a) Codificación MATLAB

```
>> f= inline ('-32.1740+32.1740/(352/3)^1.5 *abs(v)^1.5','t','v');
>> [t,y]= rungekutta (f,0,10,0,1000);
>> plot (t, y*60^2/5280)
>> flibre = inline ('-32.1740','t','v');
>> hold on
>> [t2,y2]= rungekutta (flibre,0,10,0,1000);
>> plot (t2, y2*60^2/5280, 'o')
>> xlabel ('tiempo en segundos'); ylabel('velocidad')
```



b)

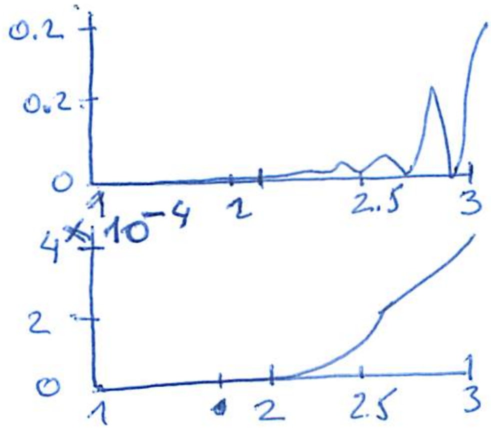
```
>> k=1;
>> While  y(k)*60^2/5280 >-60
   k=k+1;
   end
>> k       → 404
>> t(k)    → 4.03 segundos.
```

Ejercicio 5: Usar «ode45» para resolver el (PVI)

$$\begin{cases} y' = 2ty \\ y(1) = 1 \end{cases} \quad 1 \le t \le 3$$

y tomar distintas tolerancias: Sol. exacta $\equiv y(t) = e^{t^2-1}$

```
>> f=inline('2*t*y','t','y')
>> [t,y]=ode45(f,[1,3],1);   % TOL. error relativo = 10^-3
                             % TOL error absoluto = 10^-6
>> yexacta=inline('exp(t^2-1)');
>> subplot(2,1,1)
>> plot(t,abs(yexacta-y))
>> options = odeset('RelTol',1e-8)   % Otra tolerancia
                                      % para el error
>> [t2,y2]=ode45(f,[1,3],1,options);  % relativo, 10^-8
>> subplot(2,1,2)
>> plot(t2,abs(yexacta-y2))
```



[t,y]=ode45('f',[a,b],y(0),options)

PROBLEMAS DE CONTORNO

$$(1)\begin{cases} y'' = p(t)y' + q(t)y + r(t) \\ y(a) = \alpha, \; y(b) = \beta \end{cases} \qquad a \leq t \leq b$$

Teorema: Si p, q, r son continuas en $[a,b]$ y $q(t) > 0 \;\; \forall t \in [a,b]$, entonces (1) tiene solución y es única.

Método del disparo

$$(2)\begin{cases} u'' = p(t)u' + q(t)u + r(t) \\ u(a) = \alpha, \; u'(a) = 0 \end{cases} \qquad \text{(Problema de V. I.)}$$

$$(3)\begin{cases} v'' = p(t)v' + q(t)v \\ v(a) = 0, \; v'(a) = 1 \end{cases} \qquad \text{(Problema de V. I.)}$$

Si $v(b) \neq 0$ en (3) entonces la solución a (1) viene dada por:

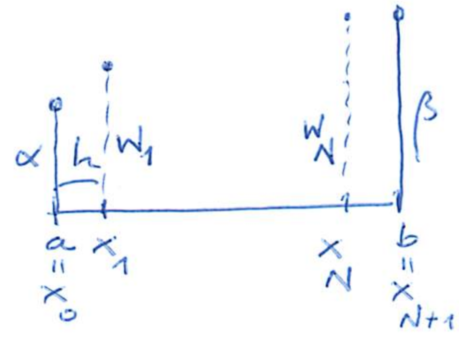
$$y(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t)$$

$$y'(t) = u'(t) + \frac{\beta - u(b)}{v(b)} v'(t)$$

Se ~~demuestra~~ muestra sin dificultad que si se cumplen las hipótesis del teorema la solución de (3) cumple $v(b) \neq 0$.

MÉTODO DE DIFERENCIAS FINITAS

$$(1)\quad \begin{cases} y'' = p(x)y' + q(x)y + r(x) \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$



Tomamos $h = \dfrac{b-a}{N+1}$, $x_i = a + ih$

Sea $y \in C^4[a,b]$, entonces

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12}y^{IV}(\xi_i)$$

$x_{i-1} < \xi_i < x_{i+1}$

$\forall i = 1, \dots, N$

$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{6}y'''(\eta_i)$$

$x_{i-1} < \eta_i < x_{i+1}$

$\forall i = 1, \dots, N$

Sustituyendo en (1) nos queda:

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p(x_i)\left[\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right] + q(x_i)y(x_i) +$$

$$+ r(x_i) - \frac{h^2}{12}\left[2p(x_i)y'''(\eta_i) - y^{IV}(\xi_i)\right] \qquad i = 1, \dots, N$$

Tomando $w_i \simeq y(x_i)$ cumpliendo:

$$w_0 = \alpha, \quad w_{N+1} = \beta$$

$$\frac{2w_i - w_{i+1} - w_{i-1}}{h^2} + p(x_i)\frac{w_{i+1} - w_{i-1}}{2h} + q(x_i)w_i = -r(x_i)$$

$i = 1, \dots, N$

Expresando el sistema lineal anterior en forma matricial, nos queda:

$$Aw = b, \text{ donde}$$

$$A = \begin{bmatrix} 2+h^2 q(x_1) & -1+\frac{h}{2}p(x_1) & 0 & \cdots & \cdots & 0 \\ -1-\frac{h}{2}p(x_2) & 2+h^2 q(x_2) & -1+\frac{h}{2}p(x_2) & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & 0 & -1-\frac{h}{2}p(x_N) & 2+\frac{h^2}{2}q(x_N) \\ 0 & \cdots & & & & \end{bmatrix}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}, \quad b = \begin{bmatrix} -\frac{h^2}{\cancel{2}} r(x_1) + \left(1+\frac{h}{2}p(x_1)\right) w_0 \\ -\frac{h^2}{\cancel{2}} r(x_2) \\ \vdots \\ -\frac{h^2}{\cancel{2}} r(x_{N-1}) \\ -\frac{h^2}{\cancel{2}} r(x_N) + \left(1-\frac{h}{2}p(x_N)\right) w_{N+1} \end{bmatrix}$$

<u>Teorema</u>: Sean p, q y r continuas en $[a,b]$. Si $q(x) \geq 0$ en $[a,b]$, entonces el sistema lineal tridiagonal, $Aw=b$, tiene una solución única siempre que $h < 2/P^*$ donde $P^* = \max_{a \leq x \leq b} |p(x)|$.

<u>Teorema</u>: En las condiciones del teorema anterior, se tiene.

$$|w_i - y(x_i)| \leq \underbrace{h^2\left(\frac{M_4 + 2P^* M_3}{12 Q_*}\right)}_{\text{Error Truncamiento}} + \underbrace{\frac{1}{h^2}\left(\frac{2\,eps}{Q_*}\right)}_{\text{Error de redondeo.}}$$

$$M_4 = \max_{a \leq x \leq b} |y^{IV}(x)|, \quad M_3 = \max_{a \leq x \leq b} |y'''(x)|; \quad P^* = \max_{a \leq x \leq b} |p(x)|$$

$$Q_* = \min_{a \leq x \leq b} |q(x)| > 0, \quad eps = \text{precisión de la máquina.}$$

Supuesto que la solución al problema es: $y \in C^4[a,b]$

SOLUCIÓN NUMÉRICA DE:

$$AX=b \quad , \quad A=(a_{ij})_{i,j=1,\dots,n} \, , \quad b=(b_i)_{i=1\dots n}$$

Algoritmo 1. (Método de Jacobi). A partir de una aproximación inicial x a la solución exacta del sistema $Ax = b$, este algoritmo proporciona una solución aproximada calculada por el método de Jacobi (ε representa la tolerancia prefijada).

(1) Para $i = 1, 2, \dots, n$, calcular

$$y_i \leftarrow \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j \right).$$

(2) Si $\|x - y\| < \varepsilon$, detener las iteraciones.

(3) Si $\|x - y\| \geq \varepsilon$, poner $x \leftarrow y$ e ir al paso (1).

Algoritmo 2. (Método de Gauss-Seidel). A partir de una aproximación inicial x a la solución exacta del sistema $Ax = b$, este algoritmo proporciona una solución aproximada calculada por el método de Gauss-Seidel (ε es la tolerancia prefijada).

(1) Para $i = 1, 2, \dots, n$, calcular

$$y_i \leftarrow \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}y_j - \sum_{j=i+1}^{n} a_{ij}x_j \right).$$

(2) Si $\|x - y\| < \varepsilon$, detener las iteraciones.

(3) Si $\|x - y\| \geq \varepsilon$, poner $x \leftarrow y$ e ir al paso (1).

Algoritmo 3. (Método SOR). Sea A una matriz cuadrada para la que el método SOR con factor ω (que, en general, será ω_{opt}) es convergente. Este algoritmo proporciona, a partir de una aproximación inicial x, una solución aproximada del sistema $Ax = b$ calculada con dicho método (ε es la tolerancia prefijada).

(1) Para $i = 1, 2, \dots, n$, calcular

$$y_i \leftarrow x_i + \frac{\omega}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij}y_j - \sum_{j=i}^{n} a_{ij}x_j \right).$$

(2) Si $\|x - y\| < \varepsilon$, detener las iteraciones.

(3) Si $\|x - y\| \geq \varepsilon$, poner $x \leftarrow y$ e ir al paso (1).

ECUACIÓN DEL CALOR (NO HOMOGÉNEA) (MÉTODO PROGRESIVO)

Sea el problema:

$$
(1)\begin{cases}
\dfrac{\partial u}{\partial t}(x,t) = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) & 0<x<L,\ t>0 \\
u(0,t) = a(t)\ ;\ u(L,t) = b(t) & t>0 \\
u(x,0) = f(x) & 0\leq x \leq L
\end{cases}
$$

Tomamos $h = L/m$, $k>0$. Los puntos de la malla son (x_i, t_j) donde $x_i = ih$, $i=0,\dots,m$ y $t_j = jk$ $j=0,1,\dots$

Usando en (1) las fórmulas.

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)$$

donde $\mu_j \in (t_j, t_{j+1})$.

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_i+h, t_j) - 2u(x_i,t_j) + u(x_i-h, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)$$

donde $\xi_i \in (x_{i-1}, x_{i+1})$,

Eliminando términos pequeños:

$$\frac{w_{i,j+1} - w_{i,j}}{k} = \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + F_{i,j}$$

donde $w_{i,j} \simeq u(x_i, y_j)$ y $F_{i,j} = F(x_i, y_j)$

$$w_{i,j+1} = (1-2\lambda)w_{i,j} + \lambda w_{i+1,j} + \lambda w_{i-1,j} + kF_{i,j}$$

con $\lambda = \dfrac{\alpha^2 k}{h^2}$, donde $w_{0,j} = a(t_j)$, $w_{m,j} = b(t_j)$

y $w_{i,0} = f(x_i)$ son conocidos.

Expresándolo en forma matricial.

$$(w_{1,0}; w_{2,0}; \dots; w_{m-1,0}) = (f(x_1); f(x_2); \dots; f(x_{m-1}))$$

$$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & \lambda & 1-2\lambda & \lambda \\ 0 & \cdots 0 & 0 & \cdots \lambda & 1-2\lambda \end{bmatrix}}_{\text{MATRIZ TRIDIAGONAL} = A} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} +$$

$$+ \underbrace{\begin{bmatrix} kF_{1,j} + \lambda\, a(t_j) \\ kF_{2,j} \\ \vdots \\ kF_{m-2,j} \\ kF_{m-1,j} + \lambda\, b(t_j) \end{bmatrix}}_{= b_j}$$

con $\lambda = \dfrac{k\alpha^2}{h^2}$

$$\boxed{\begin{aligned} W_0 &= (f(x_i))_{i=1,\dots,m-1} \\ W_{j+1} &= AW_j + b_j \end{aligned}}$$

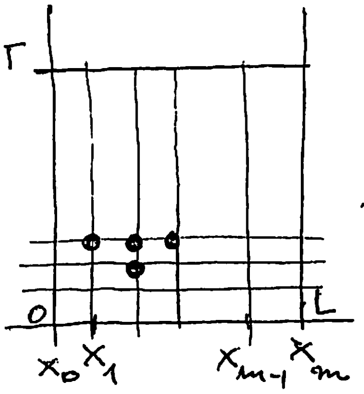
$$W_j = (w_{i,j})_{i=1,\dots,m-1}$$

CONDICIÓN DE ESTABILIDAD: $\lambda \leq 1/2$

Error $\simeq O(k+h^2)$

M. Implícito (Condiciones de Contorno tipo Dirichlet)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x;t) = c^2 \dfrac{\partial^2 u}{\partial t}(x;t) + F(x;t) & 0<x<L, \quad t>0 \quad (1) \\ u(x,0) = f(x) & 0\le x\le L \quad (\text{Cond. inicial}) \\ u(0,t) = a(t)\ ;\ u(L,t) = b(t) & t>0 \quad (\text{Cond. Contorno } \underline{\text{Dirichlet}}) \end{cases}$$



$h = \frac{L}{m}$; $k = T/N$ $\quad \Omega = \{(x_i, t_j) \,/\, i = 1,\dots,m-1;\ j = 1,\dots,N\}$

$x_i = ih$; $t_j = kj$ $\quad$ (Rejilla interior)

$i = 0,\to m \qquad j = 0,\to N$

$$(*)\begin{cases} u_t(x, t+k) = \dfrac{u(x, t+k) - u(x,t)}{k} + O(k) \\ u_{xx}(x,t+k) = \dfrac{u(x+h, t+k) - 2u(x,t+k) + u(x-h, t+k)}{h^2} + O(h^2) \end{cases}$$

Usando $(*)$ en (1) para $(x, t+k) = (x_i, t_j + k)$ $\quad i = 1,\dots,m-1;\ j = 0,\dots,N-1$

y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = c^2 \frac{W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1}}{h^2} + F_{i,j+1}$$

con $W_{i,j} \simeq u(x_i, t_j)$ y $F_{i,j} = F(x_i, t_j)$.

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + kF_{i,j+1} \quad (2)$$

con $\lambda = \dfrac{c^2 k}{h^2}$.

Cond. iniciales $W_{i,0} = f(x_i)$; $W_{0,j} = a(t_j)$; $W_{m,j} = b(t_j)$ $\quad i = 0,\dots,m;\ j = 1,\dots,N$

En forma matricial. (2), queda:

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & \ddots & \ddots & \ddots & \\ & \bigcirc & \ddots & \ddots & -\lambda \\ & & & -\lambda & 1+2\lambda \end{bmatrix} \underbrace{\begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{1,j+1} + \lambda a(t_{j+1}) \\ kF_{2,j+1} \\ \vdots \\ kF_{m-2,j+1} \\ kF_{m-1,j+1} + \lambda b(t_{j+1}) \end{bmatrix}}_{c_j}$$

Algoritmo (DT)

$\to W_0 = (f(x_i))_{i=1}^{m-1}$ dato

$$AW_{j+1} = W_j + c_j \qquad j = 0, 1, \dots, N-1$$

Crank-Nicolson (Cond. Contorno tipo Dirichlet)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x;t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x;t) + F(x;t) & 0<x<L,\ t>0 \quad (1) \\ u(x,0) = f(x) \quad 0 \le x \le L & (\text{C.I}) \\ u(0,t) = a(t);\ u(L,t) = b(t) \quad t>0 & (\text{C.Co. Dirichlet}) \end{cases}$$

Fórmulas de derivadas

$$\begin{cases} f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \\ f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \dots \end{cases}$$, restando queda:

(A) $\boxed{f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} + O(h^2)}$; y sumando queda:

(B) $\boxed{f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)}$ y (C) $\boxed{f(x) = \dfrac{f(x+h) + f(x-h)}{2} + O(h^2)}$

Por tanto, usando (A) obtenemos: $\boxed{u_t\left(x, t+\frac{k}{2}\right) = \dfrac{u(x,t+k) - u(x,t)}{k} + O(k^2)}$

y usando (B), obtenemos: $u_{xx}(x,t) = \dfrac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)$

$u_{xx}(x,t+k) = \dfrac{u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)}{h^2} + O(h^2)$.

Usando (C) se tiene $\boxed{u_{xx}\left(x,t+\frac{k}{2}\right) = \dfrac{u_{xx}(x,t+k) + u_{xx}(x,t)}{2}}$

Tomando en (1) los nodos $(x,t) = \left(x_i, t_j + \frac{k}{2}\right)$ $\quad i = 1, \dots, m-1$; $j = 0, \dots, N-1$

$$\frac{\partial}{\partial t} u\left(x_i, t_j + \frac{k}{2}\right) = c^2 \frac{\partial^2 u}{\partial x^2}\left(x_i, t_j + \frac{k}{2}\right) + F\left(x_i, t_j + \frac{k}{2}\right).$$

Usando las fórmulas obtenidas anteriormente y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = \frac{c^2}{2h^2}\left[W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1} + W_{i+1,j} - 2W_{i,j} + W_{i-1,j}\right] + F_{i,j+\frac{1}{2}}$$

$$-\frac{\lambda}{2} W_{i-1,j+1} + (1+\lambda) W_{i,j+1} - \frac{\lambda}{2} W_{i+1,j+1} = \frac{\lambda}{2} W_{i-1,j} + (1-\lambda) W_{i,j} + \frac{\lambda}{2} W_{i+1,j} + k F_{i,j+\frac{1}{2}}$$

$$\begin{bmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-\lambda & \lambda/2 & & \\ \lambda/2 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix} + \begin{bmatrix} kF_{1,j+\frac{1}{2}} + \frac{\lambda}{2}(*) \\ kF_{2,j+\frac{1}{2}} \\ \vdots \\ kF_{m-1,j+\frac{1}{2}} + \frac{\lambda}{2}(**) \end{bmatrix}$$

$(*) = (a(t_{j+1}) + a(t_j))$, $(**) = (b(t_{j+1}) + b(t_j))$

Método Implícito (Cond. Contorno tipo Neumann)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x;t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x;t) + F(x;t) & 0<x<L\ ,\ t>0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad (\text{Cond. inicial}) \\ \dfrac{\partial u}{\partial x}(0,t) = a(t)\ ;\ \dfrac{\partial u}{\partial x}(L,t) = b(t) & t>0 \quad (\text{Cond. Cont. Neumann}) \end{cases}$$

Discretización de las Cond. Contorno. $\left(\dfrac{\partial u}{\partial x}(x,t) = \dfrac{u(x+h,t) - u(x-h,t)}{2h} + O(h^2 \right.$

$$\frac{W_{1,j} - W_{-1,j}}{2h} = a(t_j) \quad y \quad \frac{W_{m+1,j} - W_{m-1,j}}{2h} = b(t_j) \quad (3)$$

$$W_{i,j} \simeq u(x_i, t_j) \quad \left[\!\left[\ W_{-1,j} \simeq u(x_{-1}, t_j)\ ;\ W_{m+1,j} \simeq u(x_{m+1}, t_j)\ \right]\!\right]$$

$$x_{-1} = (-1)h \qquad x_{m+1} = (m+1)h$$

Por tanto: $W_{-1,j} = \underset{\text{anterior}}{W_{1,j}} - 2h\, a(t_j)$ y $W_{m+1,j} = W_{m-1,j} + 2h\, b(t_j)$ (3)

Usando la discretización de (1) para $(x_i, t_j + k)$ $i = 0, \dots, m$; $j = 0, \dots, N-1$

obtenemos:

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + k F_{i,j+1} \qquad \begin{matrix} i = 0, \dots, m \\ j = 0, \dots, N-1 \end{matrix}$$

sustituyendo $W_{-1,j+1}$ y $W_{m+1,j+1}$ nos queda.

$$\underbrace{\begin{bmatrix} 1+2\lambda & -2\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & -\lambda & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda \\ & & & -2\lambda & 1+2\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} W_{0,j+1} \\ W_{1,j+1} \\ \vdots \\ W_{m-1,j+1} \\ W_{m,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{0,j} \\ W_{1,j} \\ \vdots \\ W_{m-1,j} \\ W_{m,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{0,j+1} - 2\lambda h\, a(t_{j+1}) \\ kF_{1,j+1} \\ \vdots \\ kF_{m-1,j+1} \\ kF_{m,j+1} + 2\lambda h\, b(t_{j+1}) \end{bmatrix}}_{C_j}$$

$$\text{I)} \begin{cases} W_0 = \left(f(x_i)\right)_{i=0}^{m} \\ A W_{j+1} = W_j + C_j \qquad j = 0, 1, \dots, N-1 \end{cases}$$

(1)

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}$$

$$y(1)=1,\ y(2)=2$$

SOL EXACTA:

$$y = c_1 x - \frac{c_2}{x} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$$

DISPARO (1)

x_i	$u_{1,i}$	$v_{1,i}$	w_i	$y(x_i)$	$\|y(x_i) - w_i\|$
1.0	1.00000000	0.00000000	1.00000000	1.00000000	—
1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
1.2	1.03245472	0.16851175	1.18708471	1.18708484	1.34×10^{-7}
1.3	1.06674375	0.23608704	1.28338227	1.28338236	9.78×10^{-8}
1.4	1.10928795	0.29659067	1.38144589	1.38144595	6.02×10^{-8}
1.5	1.15830000	0.35184379	1.48115939	1.48115942	3.06×10^{-8}
1.6	1.21248372	0.40311695	1.58239245	1.58239246	1.08×10^{-8}
1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}
1.9	1.39750618	0.54098928	1.89392951	1.89392951	4.41×10^{-9}
2.0	1.46472815	0.58332538	2.00000000	2.00000000	—

$c_1 \approx 1.139207$, $c_2 \simeq -0.039207$

DIF. FINITAS (1)

x_i	w_i	$y(x_i)$	$\|w_i - y(x_i)\|$
1.0	1.00000000	1.00000000	—
1.1	1.09260052	1.09262930	2.88×10^{-5}
1.2	1.18704313	1.18708484	4.17×10^{-5}
1.3	1.28333687	1.28338236	4.55×10^{-5}
1.4	1.38140205	1.38144595	4.39×10^{-5}
1.5	1.48112026	1.48115942	3.92×10^{-5}
1.6	1.58235990	1.58239246	3.26×10^{-5}
1.7	1.68498902	1.68501396	2.49×10^{-5}
1.8	1.78888175	1.78889853	1.68×10^{-5}
1.9	1.89392110	1.89392951	8.41×10^{-6}
2.0	2.00000000	2.00000000	—

(2)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) & 0 < x < 1,\ t > 0 \\ u(0,t)=0,\ u(1,t)=0 & t>0 \\ u(x,0) = \sin(\pi x) & 0 \le x \le 1 \end{cases}$$

SOL. EXACTA: $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$

2) M. PROGRESIVO $t=0.5$

$$\begin{cases} h=0.1;\ k=0.0005,\ \lambda=0.05 \\ h=0.1,\ k=0.01,\ \lambda=1 \end{cases}$$

x_i	$u(x_i, 0.5)$	$w_{i,1000}$ $k=0.0005$	$\|u(x_i,0.5) - w_{i,1000}\|$	$w_{i,50}$ $k=0.01$	$\|u(x_i,0.5) - w_{i,50}\|$
0.0	0	0	—	0	—
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0	—	0	—

(2) M. REGRESIVO: $h=0.1$, $k=0.01$

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\|w_{i,50} - u(x_i, 0.5)\|$
0.0	0	0	—
0.1	0.00289802	0.00222241	6.756×10^{-4}
0.2	0.00551236	0.00422728	1.285×10^{-3}
0.3	0.00758711	0.00581836	1.769×10^{-3}
0.4	0.00891918	0.00683989	2.079×10^{-3}
0.5	0.00937818	0.00719188	2.186×10^{-3}
0.6	0.00891918	0.00683989	2.079×10^{-3}
0.7	0.00758711	0.00581836	1.769×10^{-3}
0.8	0.00551236	0.00422728	1.285×10^{-3}
0.9	0.00289802	0.00222241	6.756×10^{-4}
1.0	0	0	—

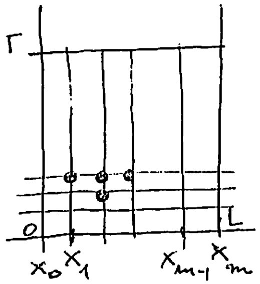
(2) CRANK-NICOLSON

$h=0.1$, $K=0.01$ $(\lambda=1)$

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\|w_{i,50} - u(x_i, 0.5)\|$
0.0	0	0	—
0.1	0.00230512	0.00222241	8.271×10^{-5}
0.2	0.00438461	0.00422728	1.573×10^{-4}
0.3	0.00603489	0.00581836	2.165×10^{-4}
0.4	0.00709444	0.00683989	2.546×10^{-4}
0.5	0.00745954	0.00719188	2.677×10^{-4}
0.6	0.00709444	0.00683989	2.546×10^{-4}
0.7	0.00603489	0.00581836	2.165×10^{-4}
0.8	0.00438461	0.00422728	1.573×10^{-4}
0.9	0.00230512	0.00222241	8.271×10^{-5}
1.0	0	0	—

M. Implícito (Condiciones de Contorno tipo Dirichlet)

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial t}(x,t) + F(x,t) & 0 < x < L, \ t > 0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{(Cond. inicial)} \\ u(0,t) = a(t) \ ; \ u(L,t) = b(t) & t > 0 \quad \text{(Cond. Contorno Dirichlet)} \end{cases}$$



$h = \frac{L}{m}$; $k = T/N$ $\quad \Omega = \{(x_i, t_j) \ / \ i = 1, \dots, m-1 ; \ j = 1, \dots, N\}$

$x_i = ih$; $t_j = kj$ $\quad$ (Rejilla interior

$i = 0, \dots, m \quad j = 0, \dots, N$

$$(*) \begin{cases} u_t(x, t+k) = \frac{u(x, t+k) - u(x,t)}{k} + O(k) \\ u_{xx}(x, t+k) = \frac{u(x+h, t+k) - 2u(x, t+k) + u(x-h, t+k)}{h^2} + O(\end{cases}$$

Usando $(*)$ en (1) para $(x, t+k) = (x_i, t_j + k)$ $\quad i = 1, \dots, m-1$; $j = 0, \dots, N-1$

y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = c^2 \frac{W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1}}{h^2} + F_{i,j+1}$$

con $W_{i,j} \simeq u(x_i, t_j)$ y $F_{i,j} = F(x_i, t_j)$.

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + k F_{i,j+1} \quad (2)$$

con $\lambda = \frac{c^2 k}{h^2}$.

Cond. iniciales $W_{i,0} = f(x_i)$; $W_{0,j} = a(t_j)$; $W_{m,j} = b(t_j)$ $\quad i = 0, \dots, m$; $j = 1, \dots, N$

En forma matricial. (2), queda:

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda \\ 0 & & & -\lambda & 1+2\lambda \end{bmatrix} \underbrace{\begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{1,j+1} + \lambda a(t_{j+1}) \\ kF_{2,j+1} \\ \vdots \\ kF_{m-2,j+1} \\ kF_{m-1,j+1} + \lambda b(t_{j+1}) \end{bmatrix}}_{c_j}$$

Algoritmo (EI)

$$\to W_0 = (f(x_i))_{i=1}^{m-1} \quad \text{dato}$$

$$A W_{j+1} = W_j + c_j \qquad j = 0, 1, \dots, N-1$$

Método Implícito (Cond. Contorno tipo Neumann)

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) & 0 < x < L, \ t > 0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{(Cond. inicial)} \\ \frac{\partial u}{\partial x}(0,t) = a(t) \ ; \ \frac{\partial u}{\partial x}(L,t) = b(t) & t > 0 \quad \text{(Cond. Cont. Neumann)} \end{cases}$$

Discretización de las Cond. Contorno. $\left(\frac{\partial u}{\partial x}(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + O \right.$

$$\frac{W_{1,j} - W_{-1,j}}{2h} = a(t_j) \quad y \quad \frac{W_{m+1,j} - W_{m-1,j}}{2h} = b(t_j) \quad (3)$$

$W_{i,j} \simeq u(x_i, t_j)$ $\quad \left[W_{-1,j} \simeq u(x_{-1}, t_j) \right.$; $W_{m+1,j} \simeq u(x_{m+1}, t_j)$

$x_{-1} = (-1)h \qquad x_{m+1} = (m+1)h$

Por tanto: $W_{-1,j} = W_{1,j} - 2h\, a(t_j)$ y $W_{m+1,j} = W_{m-1,j} + 2h\, b(t_j)$

Usando la discretización de (1) para $(x_i, t_j + k)$ $\quad i = 0, \dots, m$; $j = 0, \dots, N-1$

obtenemos:

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + k F_{i,j+1} \qquad i = 0, \dots, m \quad j = 0, \dots, N-1$$

sustituyendo $W_{-1,j+1}$ y $W_{m+1,j+1}$ nos queda.

$$\underbrace{\begin{bmatrix} 1+2\lambda & -2\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & -\lambda & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda \\ 0 & & & -2\lambda & 1+2\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} W_{0,j+1} \\ W_{1,j+1} \\ \vdots \\ W_{m-1,j+1} \\ W_{m,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{0,j} \\ W_{1,j} \\ \vdots \\ W_{m-1,j} \\ W_{m,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{0,j+1} - 2\lambda h\, a(t_j \\ kF_{1,j+1} \\ \vdots \\ kF_{m-1,j+1} \\ kF_{m,j+1} + 2\lambda h\, b(t \end{bmatrix}}_{c_j}$$

NI)
$$W_0 = (f(x_i))_{i=0}^{m}$$
$$A W_{j+1} = W_j + c_j \qquad j = 0, 1, \dots, N-1$$

Crank-Nicolson (Cond. Contorno tipo Dirichlet)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) & 0<x<L,\ t>0 \quad (1) \\ u(x,0) = f(x) \qquad 0 \le x \le L & (C\,I) \\ u(0,t) = a(t);\ u(L,t) = b(t) \qquad t>0 & (\text{C.Co. Dirichlet}) \end{cases}$$

Fórmulas de derivadas

$$\begin{cases} f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \\ f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \cdots \end{cases}$$, restando queda:

(A) $\boxed{f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} + O(h^2)}$; y sumando queda:

(B) $\boxed{f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)}$ y (C) $\boxed{f(x) = \dfrac{f(x+h)+f(x-h)}{2} + O}$

Por tanto, usando (A) obtenemos: $\boxed{u_t(x, t+\frac{k}{2}) = \dfrac{u(x,t+k) - u(x,t)}{k} + O(k^2)}$

y usando (B), obtenemos: $u_{xx}(x,t) = \dfrac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)$,

y $u_{xx}(x, t+k) = \dfrac{u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)}{h^2} + O(h^2)$.

Usando (C) se tiene $\boxed{u_{xx}(x, t+\frac{k}{2}) = \dfrac{u_{xx}(x,t+k) + u_{xx}(x,t)}{2}}$

Tomando en (1) los nodos $(x,t) = (x_i, t_j + \frac{k}{2})$ $\quad i = 1, \dots, m-1$; $j = 0, \dots, N-1$

$$\frac{\partial}{\partial t} u(x_i, t_j + \tfrac{k}{2}) = c^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j + \tfrac{k}{2}) + F(x_i, t_j + \tfrac{k}{2}).$$

Usando las fórmulas obtenidas anteriormente y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = \frac{c^2}{2h^2}\left[W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1} + W_{i+1,j} - 2W_{i,j} + W_{i-1,j}\right] + F_{i,j}$$

$$-\frac{\lambda}{2} W_{i-1,j+1} + (1+\lambda) W_{i,j+1} - \frac{\lambda}{2} W_{i+1,j+1} = \frac{\lambda}{2} W_{i-1,j} + (1-\lambda) W_{i,j} + \frac{\lambda}{2} W_{i+1,j} + k F_{i,j+\frac{1}{2}}$$

$$\begin{bmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-\lambda & \lambda/2 & \\ \lambda/2 & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix} + \begin{bmatrix} kF_{1,j+\frac{1}{2}} + \frac{\lambda}{2}(*) \\ kF_{2,j+\frac{1}{2}} \\ \vdots \\ kF_{m-1,j+\frac{1}{2}} + \frac{\lambda}{2}(**) \end{bmatrix}$$

$(**) = (b(t_{j+1}) + b(t_j))$

Crank-Nicolson (Cond. Contorno de tipo Neumann)

C.Co. Neumann $\quad \dfrac{\partial u}{\partial x}(0,t) = a(t)$; $\dfrac{\partial u}{\partial x}(L,t) = b(t) \quad t>0$

y discretizando: $W_{-1,j} = W_{1,j} - 2h\,a(t_j)$; $W_{m+1,j} = W_{m-1,j} + 2h\,b(t_j)$

La discretiz. de la EDP es

$$-\frac{\lambda}{2} W_{i-1,j+1} + (1+\lambda) W_{i,j+1} - \frac{\lambda}{2} W_{i+1,j+1} = \frac{\lambda}{2} W_{i-1,j} + (1-\lambda) W_{i,j} + \frac{\lambda}{2} W_{i+1,j} + k F_{i,j+\frac{1}{2}}$$

$i = 0, 1, \dots, m$; $j = 0, 1, \dots, N-1$

En forma matricial se expresa:

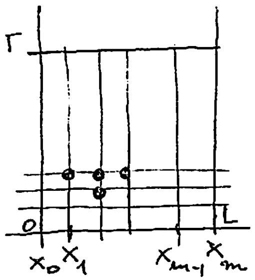
$$\underbrace{\begin{bmatrix} 1+\lambda & -\lambda & & \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \\ & \ddots & \ddots & -\lambda/2 \\ & & -\lambda & 1+\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} W_{0,j+1} \\ W_{1,j+1} \\ \vdots \\ W_{m-1,j+1} \\ W_{m,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} 1-\lambda & \lambda & & \\ \lambda/2 & 1-\lambda & \lambda/2 & \\ & \ddots & \ddots & \lambda/2 \\ & & \lambda & 1-\lambda \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} W_{0,j} \\ W_{1,j} \\ \vdots \\ W_{m-1,j} \\ W_{m,j} \end{bmatrix}}_{W_j}$$

$$+ \underbrace{\begin{bmatrix} kF_{0,j+1/2} - \lambda h\left[a(t_{j+1}) + a(t_j)\right] \\ kF_{1,j+1/2} \\ \vdots \\ kF_{m-1,j+1/2} \\ kF_{m,j+1/2} + \lambda h\left[b(t_{j+1}) + b(t_j)\right] \end{bmatrix}}_{C_j}$$

Neumann Crank-Nicolson
$$\begin{cases} W_0 = \left(f(x_i)\right)_{i=0}^{m} \quad \text{dato} \\ A W_{j+1} = B W_j + C_j \qquad j = 0, 1, \dots, N-1 \end{cases}$$

M. Implícito (Condiciones de Contorno tipo Dirichlet)

$$\begin{cases} \frac{\partial u}{\partial t}(x;t) = c^2 \frac{\partial^2 u}{\partial t}(x;t) + F(x;t) & 0 < x < L,\ t > 0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{(Cond. inicial)} \\ u(0,t) = a(t);\ u(L,t) = b(t) & t > 0 \quad \text{(Cond. Contorno Dirichlet)} \end{cases}$$



$h = \frac{L}{m}$; $k = T/N$ $\quad \Omega = \{(x_i, t_j) / i = 1, \dots, m-1;\ j = 1, \dots, N\}$

$x_i = ih$; $t_j = kj$ $\quad$ (Rejilla inferior)

$i = 0, \dots, m$ $\quad j = 0, \dots, N$

$$(*)\begin{cases} u_t(x, t+k) = \frac{u(x, t+k) - u(x,t)}{k} + O(k) \\ u_{xx}(x, t+k) = \frac{u(x+h, t+k) - 2u(x, t+k) + u(x-h, t+k)}{h^2} + O(\end{cases}$$

Usando $(*)$ en (1) para $(x, t+k) = (x_i, t_j + k)$ $\quad i = 1, \dots, m-1;\ j = 0, \dots, N-1$
y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = c^2 \frac{W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1}}{h^2} + F_{i,j+1}$$

con $W_{i,j} \simeq u(x_i, t_j)$ y $F_{i,j} = F(x_i, t_j)$.

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + k F_{i,j+1} \quad (2)$$

con $\lambda = \frac{c^2 k}{h^2}$.

Cond. iniciales $W_{i,0} = f(x_i)$; $W_{0,j} = a(t_j)$; $W_{m,j} = b(t_j)$ $\quad i = 0, \dots, m;\ j = 1, \dots, N$

En forma matricial. (2), queda:

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & \ddots & \ddots & \ddots & \\ & 0 & & & -\lambda \\ & & & -\lambda & 1+2\lambda \end{bmatrix} \underbrace{\begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{1,j+1} + \lambda a(t_{j+1}) \\ kF_{2,j+1} \\ \vdots \\ kF_{m-2,j+1} \\ kF_{m-1,j+1} + \lambda b(t_{j+1}) \end{bmatrix}}_{c_j}$$

Algoritmo

$$\begin{cases} W_0 = (f(x_i))_{i=1}^{m-1} \quad \text{dato} \\ AW = W_i + c_i \qquad i = 0, 1, \dots, N-1 \end{cases}$$

Método Implícito (Cond. Contorno tipo Neumann)

$$\begin{cases} \frac{\partial u}{\partial t}(x;t) = c^2 \frac{\partial^2 u}{\partial x^2}(x;t) + F(x;t) & 0 < x < L,\ t > 0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{(Cond. inicial)} \\ \frac{\partial u}{\partial x}(0,t) = a(t);\ \frac{\partial u}{\partial x}(L,t) = b(t) & t > 0 \quad \text{(Cond. Cont. Neumann)} \end{cases}$$

Discretización de las Cond. Contorno. $\left(\frac{\partial u}{\partial x}(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(\right.$

$$\frac{W_{1,j} - W_{-1,j}}{2h} = a(t_j) \quad y \quad \frac{W_{m+1,j} - W_{m-1,j}}{2h} = b(t_j) \quad (3)$$

$W_{i,j} \simeq u(x_i, t_j)$ $\quad$ $W_{-1,j} \simeq u(x_{-1}, t_j)$; $W_{m+1,j} \simeq u(x_{m+1}, t_j)$

$x_{-1} = (-1)h$ $\quad$ $x_{m+1} = (m+1)h$

Por tanto: $W_{-1,j} = \underset{\text{anterior}}{W_{1,j}} - 2h\,a(t_j)$ y $W_{m+1,j} = W_{m-1,j} + 2h\,b(t_j)$ (3

Usando la discretización de (1) para $(x_i, t_j + k)$ $\quad i = 0, \dots, m;\ j = 0, \dots, N-1$

obtenemos:

$$-\lambda W_{i-1,j+1} + (1+2\lambda) W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + k F_{i,j+1} \qquad i = 0, \dots, m;\ j = 0, \dots, N-1$$

sustituyendo $W_{-1,j+1}$ y $W_{m+1,j+1}$ nos queda.

$$\underbrace{\begin{bmatrix} 1+2\lambda & -2\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & & \vdots \\ & -\lambda & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda \\ 0 & & & -2\lambda & 1+2\lambda \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} W_{0,j+1} \\ W_{1,j+1} \\ \vdots \\ W_{m-1,j+1} \\ W_{m,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} W_{0,j} \\ W_{1,j} \\ \vdots \\ W_{m-1,j} \\ W_{m,j} \end{bmatrix}}_{W_j} + \underbrace{\begin{bmatrix} kF_{0,j+1} - 2\lambda h\,a(t_{j+1}) \\ kF_{1,j+1} \\ \vdots \\ kF_{m-1,j+1} \\ kF_{m,j+1} + 2\lambda h\,b(t_{j+1}) \end{bmatrix}}_{c_j}$$

$$NI) \begin{cases} W_0 = (f(x_i))_{i=0}^{m} \\ AW_{j+1} = W_j + c_j \qquad j = 0, 1, \dots, N-1 \end{cases}$$

Crank-Nicolson (Cond. Contorno tipo Dirichlet)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x;t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x;t) + F(x;t) & 0<x<L,\ t>0 \quad (1) \\ u(x,0) = f(x) & 0 \le x \le L \quad (C.I) \\ u(0,t) = a(t);\ u(L,t) = b(t) & t>0 \quad (C.Co.\ Dirichlet) \end{cases}$$

Fórmulas de derivadas

$$\begin{cases} f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots \\ f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \dots \end{cases}$$, restando queda:

(A) $$\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)}$$; y sumando queda:

(B) $$\boxed{f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)}$$ y (C) $$\boxed{f(x) = \frac{f(x+h) + f(x-h)}{2} + O(h^2)}$$

Por tanto, usando (A) obtenemos: $$\boxed{u_t\left(x, t+\frac{k}{2}\right) = \frac{u(x,t+k) - u(x,t)}{k} + O(k^2)}$$

y usando (B), obtenemos: $$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2)$$

$$u_{xx}(x,t+k) = \frac{u(x+h,t+k) - 2u(x,t+k) + u(x-h,t+k)}{h^2} + O(h^2).$$

Usando (C) se tiene $$\boxed{u_{xx}\left(x,t+\frac{k}{2}\right) = \frac{u_{xx}(x,t+k) + u_{xx}(x,t)}{2}}$$

Tomando en (1) los nodos $(x,t) = \left(x_i, t_j + \frac{k}{2}\right)$ $\quad i = 1, \dots, m-1$; $j = 0, \dots, N-1$

$$\frac{\partial}{\partial t} u\left(x_i, t_j + \frac{k}{2}\right) = c^2 \frac{\partial^2 u}{\partial x^2}\left(x_i, t_j + \frac{k}{2}\right) + F\left(x_i, t_j + \frac{k}{2}\right).$$

Usando las fórmulas obtenidas anteriormente y eliminando los errores de truncamiento:

$$\frac{W_{i,j+1} - W_{i,j}}{k} = \frac{c^2}{2h^2}\left[W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1} + W_{i+1,j} - 2W_{i,j} + W_{i-1,j}\right] + F_{i,j+\frac{1}{2}}$$

$$-\frac{\lambda}{2} W_{i-1,j+1} + (1+\lambda) W_{i,j+1} - \frac{\lambda}{2} W_{i+1,j+1} = \frac{\lambda}{2} W_{i-1,j} + (1-\lambda) W_{i,j} + \frac{\lambda}{2} W_{i+1,j} + k F_{i,j+\frac{1}{2}}$$

$$\begin{bmatrix} 1+\lambda & -\lambda/2 & 0 & \cdots \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j+1} \\ W_{2,j+1} \\ \vdots \\ W_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 1-\lambda & \lambda/2 & & \\ \lambda/2 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} W_{1,j} \\ W_{2,j} \\ \vdots \\ W_{m-1,j} \end{bmatrix} + \begin{bmatrix} kF_{1,j+\frac{1}{2}} + \frac{\lambda}{2}(*) \\ kF_{2,j+\frac{1}{2}} \\ \vdots \\ kF_{m-1,j+\frac{1}{2}} + \frac{\lambda}{2}(**) \end{bmatrix}$$

$(**) = (b(t_{j+1}) + b(t_j))$

Crank-Nicolson (Cond. Contorno de tipo Neumann)

C. Co. Neumann $\dfrac{\partial u}{\partial x}(0,t)=a(t)$; $\dfrac{\partial u}{\partial x}(L,t)=b(t)$ $\quad t>0$

y discretizando: $W_{-1,j}=W_{1,j}-2h\,a(t_j)$; $W_{m+1,j}=W_{m-1,j}+2h\,b(t_j)$

La discretiz. de la EDP es

$$-\frac{\lambda}{2}W_{i-1,j+1}+(1+\lambda)W_{i,j+1}-\frac{\lambda}{2}W_{i+1,j+1}=\frac{\lambda}{2}W_{i-1,j}+(1-\lambda)W_{i,j}+\frac{\lambda}{2}W_{i+1,j}+kF_{i,j+\frac{1}{2}}$$

$i=0,1,\dots,m$

$j=0,1,\dots,N-1$

En forma matricial se expresa:

$$\underbrace{\begin{bmatrix} 1+\lambda & -\lambda & & & \\ -\lambda/2 & 1+\lambda & -\lambda/2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -\lambda/2 \\ & & & -\lambda & 1+\lambda \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} W_{0,j+1} \\ W_{1,j+1} \\ \vdots \\ W_{m-1,j+1} \\ W_{m,j+1} \end{bmatrix}}_{W_{j+1}}=\underbrace{\begin{bmatrix} 1-\lambda & \lambda & & & \\ \lambda/2 & 1-\lambda & \lambda/2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \lambda/2 \\ & & & \lambda & 1-\lambda \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} W_{0,j} \\ W_{1,j} \\ \vdots \\ W_{m-1,j} \\ W_{m,j} \end{bmatrix}}_{W_j}$$

$$+\underbrace{\begin{bmatrix} kF_{0,j+1/2}-\lambda h\left[a(t_{j+1})+a(t_j)\right] \\ kF_{1,j+1/2} \\ \vdots \\ kF_{m-1,j+1/2} \\ kF_{m,j+1/2}+\lambda h\left[b(t_{j+1})+b(t_j)\right] \end{bmatrix}}_{C_j}$$

Neumann Crank-Nicolson
$$\begin{cases} W_0=\left(f(x_i)\right)_{i=0}^{m} \quad \text{dato} \\[2ex] AW_{j+1}=BW_j+C_j \qquad j=0,1,\dots,N-1 \end{cases}$$

(1)

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}$$

$$y(1)=1,\ y(2)=2$$

SOL EXACTA:

$$y = c_1 x - \frac{c_2}{x} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$$

DISPARO (1)

x_i	$u_{1,i}$	$v_{1,i}$	w_i	$y(x_i)$	$\lvert y(x_i) - w_i\rvert$
1.0	1.00000000	0.00000000	1.00000000	1.00000000	—
1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
1.2	1.03245472	0.16851175	1.18708471	1.18708484	1.34×10^{-7}
1.3	1.06674375	0.23608704	1.28338227	1.28338236	9.78×10^{-8}
1.4	1.10928795	0.29659067	1.38144589	1.38144595	6.02×10^{-8}
1.5	1.15830000	0.35184379	1.48115939	1.48115942	3.06×10^{-8}
1.6	1.21248372	0.40311695	1.58239245	1.58239246	1.08×10^{-8}
1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}
1.9	1.39750618	0.54098928	1.89392951	1.89392951	4.41×10^{-9}
2.0	1.46472815	0.58332538	2.00000000	2.00000000	—

$c_1 \simeq 1.139207,\ c_2 \simeq -0.039207$

DIF. FINITAS (1)

x_i	w_i	$y(x_i)$	$\lvert w_i - y(x_i)\rvert$
1.0	1.00000000	1.00000000	—
1.1	1.09260052	1.09262930	2.88×10^{-5}
1.2	1.18704313	1.18708484	4.17×10^{-5}
1.3	1.28333687	1.28338236	4.55×10^{-5}
1.4	1.38140205	1.38144595	4.39×10^{-5}
1.5	1.48112026	1.48115942	3.92×10^{-5}
1.6	1.58235990	1.58239246	3.26×10^{-5}
1.7	1.68498902	1.68501396	2.49×10^{-5}
1.8	1.78888175	1.78889853	1.68×10^{-5}
1.9	1.89392110	1.89392951	8.41×10^{-6}
2.0	2.00000000	2.00000000	—

(2)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) & 0<x<1,\ t>0 \\ u(0,t)=0,\ u(1,t)=0 & t>0 \\ u(x,0) = \sin(\pi x) & 0 \le x \le 1 \end{cases}$$

SOL. EXACTA: $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

2) M. PROGRESIVO $t=0.5$ $\begin{cases} h=0.1,\ k=0.0005,\ \lambda=0.05 \\ h=0.1,\ k=0.01,\ \lambda=1 \end{cases}$

x_i	$u(x_i, 0.5)$	$w_{i,1000}$ $k=0.0005$	$\lvert u(x_i,0.5) - w_{i,1000}\rvert$	$w_{i,50}$ $k=0.01$	$\lvert u(x_i,0.5) - w_{i,50}\rvert$
0.0	0	0	—	0	—
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0	—	0	—

(2) M. REGRESIVO: $h=0.1,\ k=0.01$

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\lvert w_{i,50} - u(x_i, 0.5)\rvert$
0.0	0	0	—
0.1	0.00289802	0.00222241	6.756×10^{-4}
0.2	0.00551236	0.00422728	1.285×10^{-3}
0.3	0.00758711	0.00581836	1.769×10^{-3}
0.4	0.00891918	0.00683989	2.079×10^{-3}
0.5	0.00937818	0.00719188	2.186×10^{-3}
0.6	0.00891918	0.00683989	2.079×10^{-3}
0.7	0.00758711	0.00581836	1.769×10^{-3}
0.8	0.00551236	0.00422728	1.285×10^{-3}
0.9	0.00289802	0.00222241	6.756×10^{-4}
1.0	0	0	—

(2) CRANK-NICOLSON

$h=0.1,\ k=0.01$
$(\lambda=1)$

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\lvert w_{i,50} - u(x_i, 0.5)\rvert$
0.0	0	0	—
0.1	0.00230512	0.00222241	8.271×10^{-5}
0.2	0.00438461	0.00422728	1.573×10^{-4}
0.3	0.00603489	0.00581836	2.165×10^{-4}
0.4	0.00709444	0.00683989	2.546×10^{-4}
0.5	0.00745954	0.00719188	2.677×10^{-4}
0.6	0.00709444	0.00683989	2.546×10^{-4}
0.7	0.00603489	0.00581836	2.165×10^{-4}
0.8	0.00438461	0.00422728	1.573×10^{-4}
0.9	0.00230512	0.00222241	8.271×10^{-5}
1.0	0	0	—

Jacobi.m

```
function Y = Jacobi(A, b, TOL)
n = length(b);
X = zeros(n); Y = ones(n);
while norm(X-Y, 1) > TOL
                                  X = Y;
 for i=1:n
   Y(i) = b(i);
     for j = 1:i-1
     Y(i) = b(i) - A(i,j)*X(j);      # Gauss-Seidel Y(i) = b(i) - A(i,j)*Y(j);
     end
     for j = i+1:n
      Y(i) = b(i) - A(i,j)*X(j);
     end
   Y(i) = Y(i)/A(i,i);

end
```

Implícito Neumann - Dirichlet

$$\frac{\partial u}{\partial x}(0,t) = a(t) \; ; \; u(L,t) = b(t);$$

$$\frac{W_{1,j} - W_{-1,j}}{2h} = a(t_j) \; ; \; \boxed{W_{m,j} = b(t_j)}$$

$$\boxed{W_{-1,j} = W_{1,j} - 2h\,a(t_j)}$$

$$\frac{W_{i,j+1} - W_{i,j}}{k} = \frac{c^2}{h^2}\left[W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1}\right] + F_{i,j+1} \qquad \begin{matrix} i = 0,1,\dots,m-1 \\ j = 0,1,\dots,N-1 \end{matrix}$$

$$-\lambda W_{i-1,j+1} + (1+2\lambda)W_{i,j+1} - \lambda W_{i+1,j+1} = W_{i,j} + kF_{i,j+1} \qquad i = 0,1,\dots,m-1$$

$\boxed{i=0} \longrightarrow \boxed{-\lambda W_{-1,j+1}} + (1+2\lambda)W_{0,j+1} - \lambda W_{1,j+1} = W_{0,j} + kF_{0,j+1}$

$(1+2\lambda)W_{0,j+1} - 2\lambda W_{1,j+1} \qquad = W_{0,j} + kF_{0,j+1} - 2\lambda h\,a(t_{j+1})$

$\boxed{i=1} \rightarrow \quad -\lambda W_{0,j+1} + (1+2\lambda)W_{1,j+1} - \lambda W_{2,j+1} \qquad = W_{1,j} + kF_{1,j+1}$

$\dots$

$\boxed{i=m-2} \rightarrow \quad -\lambda W_{m-3,j+1} + (1+2\lambda)W_{m-2,j+1} - \lambda W_{m-1,j+1} \qquad = W_{m-2,j} + kF_{m-2,j+1}$

$\boxed{i=m-1} \longrightarrow \quad -\lambda W_{m-2,j+1} + (1+2\lambda)W_{m-1,j+1} - \boxed{\lambda W_{m,j+1}} \qquad = W_{m-1,j} + kF_{m-1,j+1}$

$-\lambda W_{m-2,j+1} + (1+2\lambda)W_{m-1,j+1} \qquad = W_{m-1,j} + kF_{m-1,j+1} + \lambda b(t_{j+1})$

(N-D Imp)

$$\left[\begin{array}{l} W_0 = \left(f(x_i)\right)_{i=0}^{m-1} \\ \\ AW_{j+1} = W_j + C_j \end{array}\right. , \text{ donde } A = \begin{bmatrix} 1+2\lambda & \boxed{-2\lambda} & & \\ -\lambda & 1+2\lambda & -\lambda & 0 \\ & \ddots & \ddots & -\lambda \\ 0 & & -\lambda & 1+2\lambda \end{bmatrix}$$

$$\boxed{u(x,t) = \sin x + \sin t}$$

$u_t = \cos t \qquad u_{xx} = -\sin x \qquad u_x = \cos x$

$$\begin{cases} u_t = u_{xx} + (\sin x + \cos t) & 0 < x < \pi \ , \ t > 0 \\ u(x,0) = \sin x \\ \dfrac{\partial u}{\partial x}(0,t) = 1 \ ; \ u(\pi, t) = \sin t, \ \cancel{u(0,t) = \sin t} \end{cases}$$

Calculo de $u(x,T)$, para $x = x_0, x_1, \dots, x_{m-1}$;

Algoritmo
$$\left[\begin{array}{l} W_0 = (f(x_i))_{i=0}^{m-1} \\ A W_{j+1} = W_j + c_j \end{array}\right.$$

$x_i = ih$, $h = \pi/m$

donde $A = \begin{bmatrix} 1+2\lambda & -2\lambda & & 0 \\ -\lambda & 1+2\lambda & -\lambda & \\ & \ddots & \ddots & -\lambda \\ 0 & & -\lambda & 1+2\lambda \end{bmatrix}$

$$c_j = \begin{bmatrix} kF_{0,j+1} - 2\lambda h\, a(t_{j+1}) \\ kF_{1,j+1} \\ \vdots \\ kF_{m-2,j+1} \\ kF_{m-1,j+1} + \lambda b(t_{j+1}) \end{bmatrix}$$

$j = 0, 1, \dots, N-1$

Entrada: F, f, a, b, m, N, T, L. Salida $W_N \simeq u(x_i, T)$

ImplicitoND.m

```
# Cálculo de W0 .   # Calculo A
    for j = 0 : N-1
        # Calculo cj :-> c
        # Solución del sistema de ec. lineal     W = A\(W+c)
    end
    return W
```

$\boxed{W = A\backslash(W+c)}$ ↳ Jacobi $(A, W+c, 10^{-4})$

Mejor usar Jacobi o Gauss-Seidel para ste módulo

ECUACIÓN DE ONDAS

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = c^2 \dfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t). & 0<x<L,\ t>0. \\[2mm] u(0,t) = a(t)\ ;\ u(L,t) = b(t). & t>0. \\[2mm] u(x,0) = f(x)\ ;\ \dfrac{\partial u}{\partial t}(x,0) = g(x). & 0 \le x \le L. \end{cases}$$

La ecuación discretizada es: $\begin{cases} h = L/m\ ; & x_i = ih\ ,\ i=0,1,\dots,m \\ k>0\ , & t_j = jk\ ,\ j=0,1,\dots \end{cases}$

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} = c^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} + F(x_i,t_j)$$

$$\begin{cases} w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2 w_{i+1,j} + \lambda^2 w_{i-1,j} - w_{i,j-1} + k^2 F_{i,j} \\ \lambda = \dfrac{ck}{h} \end{cases}$$

Esquema iterativo.

$$(E.O.)\quad \boxed{\begin{aligned} W_0 &= \left(f(x_i)\right)_{i=1}^{m-1} \\ W_1 &= \left((1-\lambda^2) f(x_i) + \frac{\lambda^2}{2}\left(f(x_{i+1}) + f(x_{i-1})\right) + k g(x_i) + \frac{k^2}{2} F(x_i,0) \right)_{i=1}^{m-1} \\ W_{j+1} &= A W_j - W_{j-1} + b_j \end{aligned}}$$

$$W_j = \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix}$$

$$A = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & & & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \lambda^2 \\ 0 & & & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} ;\ b_j = \begin{bmatrix} k^2 F_{1,j} + \lambda^2 a(t_j) \\ k^2 F_{2,j} \\ \vdots \\ k^2 F_{m-2,j} \\ k^2 F_{m-1,j} + \lambda^2 b(t_j) \end{bmatrix}$$

CONDICIONALMENTE ESTABLE , $\lambda \le 1$

Error $\approx O(k^2 + h^2)$

Ec. Ondas (Cond. Neumann)

Rejilla

$$\begin{cases} h = \dfrac{L}{m} \Rightarrow x_i = ih & i=0,\dots,m \\ k = \dfrac{T}{N} \Rightarrow t_j = jk & j = 0,\dots,N \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + F(x;t) \\ u(x,0) = f(x) \ ; \ \dfrac{\partial u}{\partial t} u(x,0) = g(x) \\ \dfrac{\partial u}{\partial x}(0,t) = a(t) \ ; \ \dfrac{\partial u}{\partial x}(L,t) = b(t) \end{cases}$$

$$\longrightarrow \begin{cases} \dfrac{w_{1,j} - w_{-1,j}}{2h} = a(t_j) \\ \dfrac{w_{m+1,j} - w_{m-1,j}}{2h} = b(t_j) \end{cases}$$

Discretiz. de Cond. Contorno

Discretización de la ecuación.

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} = \frac{c^2}{h^2}\left[w_{i-1,j} - 2w_{i,j} + w_{i+1,j}\right] + F_{i,j} \qquad \begin{matrix} i=0,\dots,m \\ j=1,\dots,N-1 \end{matrix}$$

$$\underbrace{w_{i,j+1}}_{\text{etapa } j+1 \text{ en el tiempo}} = \underbrace{\lambda^2 w_{i-1,j} + 2(1-\lambda^2) w_{i,j} + \lambda^2 w_{i+1,j}}_{\text{etapa } j \text{ en el tiempo}} - \underbrace{w_{i,j-1}}_{\text{etapa } j-1 \text{ en el tiempo}} + k^2 F_{i,j} \ ; \ \lambda = \frac{ck}{h}$$

$$\underbrace{\begin{bmatrix} w_{0,j+1} \\ w_{1,j+1} \\ \vdots \\ w_{m-1,j+1} \\ w_{m,j+1} \end{bmatrix}}_{W_{j+1}} = \underbrace{\begin{bmatrix} 2(1-\lambda^2) & \boxed{2\lambda^2} & & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \\ & \ddots & \ddots & \lambda^2 \\ 0 & & \boxed{2\lambda^2} & 2(1-\lambda^2) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} w_{0,j} \\ w_{1,j} \\ \vdots \\ w_{m-1,j} \\ w_{m,j} \end{bmatrix}}_{W_j} - \underbrace{\begin{bmatrix} w_{0,j-1} \\ w_{1,j-1} \\ \vdots \\ w_{m-1,j} \\ w_{m,j} \end{bmatrix}}_{W_{j-1}} + \underbrace{\begin{bmatrix} k^2 F_{0,j} - 2\lambda^2 h a(t_j) \\ k^2 F_{1,j} \\ \vdots \\ k^2 F_{m-1,j} \\ k^2 F_{m,j} + 2\lambda^2 h b(t_j) \end{bmatrix}}_{C_j}$$

(N Ondas)

$$W_0 = \left(f(x_i)\right)_{i=0}^{m} \ ; \ W_1 \quad \text{datos}$$

$$W_{j+1} = A W_j - W_{j-1} + C_j \qquad j = 1, 2, \dots, N-1$$

CÁLCULO DE W_1

$$\underset{W_{i,1}}{\overset{\shortparallel}{u(x_i,k)}} \overset{\text{Taylor}}{=} \underbrace{u(x_i,0)}_{f(x_i)} + \underbrace{u_t(x_i,0)}_{g(x_i)} k + O(k^2) \Rightarrow \boxed{W_1 = \left(f(x_i) + k g(x_i)\right)_{i=0}^{m}}$$

Una mejor aproximación para W_1 es:

$$u(x_i;k) = u(x_i,0) + u_t(x_i,0)k + \underbrace{\left[u_{tt}(x_i,0)\right]}_{\text{ecuación}} \frac{k^2}{2!} + O(k^3)$$

$$\left[c^2 u_{xx}(x_i,0) + F(x_i,0)\right] = c^2 f''(x_i) + F(x_i,0) \implies$$

$$\boxed{W_1 = \left(f(x_i) + k g(x_i) + \frac{k^2}{2}\left(c^2 f''(x_i) + F(x_i,0)\right)\right)_{i=0}^{m}}$$

Resolver los siguientes problemas relativos a la ecuación del calor:

1) $$u_t = u_{xx} + (\sin(x) - \sin(t)) \qquad 0 \leq x \leq 2\pi, \quad t \geq 0$$

$$\underline{u(0,t) = \cos(t)} \; ; \; \underline{u(2\pi,t) = \cos(t)}$$

$$u(x,0) = 1 + \sin(x)$$

(Condiciones de contorno tipo Dirichlet - Dirichlet.)

2) $$u_t = u_{xx} + (\sin(x) - \sin(t))$$

$$\underline{u(0,t) = \cos t} \; ; \; \underline{u_x(2\pi,t) = 1} \qquad 0 \leq x \leq 2\pi, \quad t \geq 0$$

$$u(x,0) = 1 + \sin(x)$$

(Cond. Cont. tipo Dirichlet - Neumann)

3) $$u_t = u_{xx} + (\sin(x) - \sin(t)) \qquad 0 \leq x \leq 2\pi, \quad t \geq 0$$

$$\underline{u_x(0,t) = 1} \; , \; \underline{u_x(2\pi,t) = 1}$$

$$u(x,0) = 1 + \sin(x)$$

(Cond. Cont. tipo Neumann - Neumann)

4) $$u_t = u_{xx} + (\sin(x) - \sin(t)) \qquad 0 \leq 2\pi \leq t, \quad t \geq 0$$

$$\underline{u(0,t) = \cos(t)} \; , \; \underline{2u(2\pi,t) + 3u_x(2\pi,t) = 2\cos(t) + 3}$$

$$u(x,0) = 1 + \sin(x)$$

(Cond. Cont. tipo Dirichlet - Robin)

5) $$u_t = u_{xx} + u + (-\sin(t) - \cos(t)) \qquad 0 \leq 2\pi \leq t, \quad t \geq 0$$

$$\underline{u_x(0,t) = 1} \; , \; \underline{u(2\pi,t) = \cos(t)}$$

$$u(x,0) = 1 + \sin(x)$$

6) $$u_t = 2u_{xx} + 2u_x + 2u + (-\sin(t) - 2\cos(x) - 2\cos(t))$$

$$\underline{u_x(0,t) = 1} \; , \; \underline{u(2\pi,t) + u_x(2\pi,t) = \cos t + 1}$$

$$u(x,0) = 1 + \sin(x)$$

En todos los casos la solución es: $\underline{u(x,t) = \sin(x) + \cos(t)}$