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Numerical Techniques for Solving the Black-Scholes Equation

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Declaration

I, Hadjer Medjadi, declare that this study entitled: “Numerical methods for Solving the Black-Scholes equation”, was completely undertaken by myself. All arguments developed by other authors are cited and acknowledged through complete references.

Hadjer Medjadi

Dedication

To my sweet and dear parents whose affection, love, encouragement and prayers of day and night made me get such an honour, along with all hard working and respected teachers.

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Abstract

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The Black-Scholes model (named after Fischer Black and Myron Scholes) for option valuation is a model used in financial mathematics to theoretically estimate the value of a financial option, of the European option type. However, solving the Black-Scholes equation in higher dimensions requires numerical techniques. In this Master's thesis, we propose a Chebyshev Pseudo spectral method and Euler Implicit method for pricing European call options and a comparative study of several possible configurations of these two methods. An option is a financial asset that offers the buyer the opportunity to buy or sell depending on the type of contract they hold. Each options contract will have a specific expiration date by which the holder must exercise their option, and it is either worthless or worth more than it was bought for. Black-Scholes partial differential equation presented in 1973, models the fair value of a European call option under certain market assumption. The terminal condition is derived from the difference between the stock price upon maturity and the option strike price, while the boundary conditions are derived from the put-call parity. We use the Chebyshev points as a set of points when discretizing Black-Scholes equation. Knowing that options has been priced with the use of finite differences it works as a comparison to the results of Chebyshev Pseudo- Spectral method. By approximating the initial condition with orthogonal Chebyshev polynomials and truncating the domain, the convergence rate increases significantly.

In this context, numerical experiments confirm a considerable increase in efficiency, especially for large data sets. [1] [2] [2] [3] [4]

JEL classification: C02, C63

Keywords Black-Scholes, Chebyshev Pseudo-Spectral, European Call Option, Euler Implicit.

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1 Introduction

This Master's thesis is about pricing European call options.

A European call option is a derivative contract that gives the keeper (the holder) the rights but not obligation to buy the underlying asset at the defined price at expiry date (exercise date). For an investor to profit from a call option, the stock's price, at expiry, had to be trading high enough above the strike price (the exercise price) to cover the cost of option premium. The exercise price and the expiry date are determined at the time when the option is written.

A European call option is mathematically simpler than an American call option, which is another type of option. The European call option can be only exercised on the exercise date. While the latter allows the holder to exercise the option rights at any time before and including the day of expiration, which allows the keeper to capture profit as soon as the stock price moves favourably, and to take advantage of dividend announcements as well.

The European / American classification has nothing to do with the continent of origin, they are actually terms used to describe two different types of option exercise.

A put option is a contract that gives the owner the right, but not the obligation, to sell a certain amount of the underlying asset, at a set price within a specific time. The buyer of a put option believes that the underlying stock will drop below the exercise price before the expiration date. The owner of a call option wants the asset price to rise, because the higher the stock price is, relative to the exercise price, the more the option would be worth, and the owner of a put option wants the asset price to fall as low as possible. Option pricing is all about answering the question What is the fair value to pay for an option? How much an option is worth on the market depends on the time left to the expiry date and the price of the underlying asset.

An investor can buy a call option, with exercise price 23.17 USD (US Dollar) for 0.93 USD, and the stock price rise to 25.02 USD at the expiry date, the profit will be 0.92 USD, (25.02-23.17-0.93) or 99%. If the share price goes below the exercise price, the option is worthless and the investor loses his invested money, which means (0.93 USD or 99%).

In order to estimate the value of call option at time t=0, the Black-Scholes formula is used. Fisher Black, Robert Merton and Myron Scholes, developed a formula for European option in 1973. Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their work, Black was mentioned as a contributor by the Swedish Academy, he could not share the prize with Merton and Scholes because of his death in 1995.

The black-Scholes equation is a parabolic partial differential equation, which describes the price of the option over time.

The one parameter in the model that cannot be observed using market data is the volatility of the underlying asset process. The black-Scholes call price function is strictly monotone increasing in volatility.

Hence, for each observed call price there is a unique volatility, one of the most important quantities in finance.

Implied volatility is a concept specific to options and is a prediction made by market participants of the degree to which underlying securities move in the future. implied volatility is essentially the real-time estimation of an asset's price as it trades. This provides the predicted volatility of an option's underlying asset over the entire lifespan of the option, using formulas that measure option market expectations. Higher implied volatility indicates that greater option price movement is expected in the future.

The parabolic PDE is solved in one and two dimensions using finite differences and Chebyshev Pseudo-Spectral method. Chebyshev Pseudo-Spectral method of solving partial differential equations uses global basis functions to approximate the solution function. When this solution is smooth the Spectral method can obtain high order numerical solution. We use Chebyshev polynomials as basic functions, and Chebyshev collocation points for space discretisation, in a subsequent symbolic computing program that retrieves the problem specific information from partial differential equations. A coordinate transform function maps the financial space of the equation to an appropriate computation space, and a function that applies the partial differential operator of the space transformed equation to Chebyshev polynomials at Chebyshev collocation points is created. We show numerical results obtained from pricing a European Call option using the Black-Scholes model. [5] [6] [3] [1]

2 Preliminaries

2.1 The Black-Scholes equation

The black-Scholes equation requires five variables. These inputs are volatility, The price of the underlying asset, the strike price of the option, the time until expiration of the option, and the risk-free interest rate. With these variables, it is theoretically possible for options sellers to set rational prices for the options that they are selling.

Furthermore, the model predicts that the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility. When applied to a stock option, the model incorporates the constant price variation of the stock, the time value of money, the option's strike price, and the time to the option's expiry.

It is fundamental to use a model of the option market, to derive Black-Scholes equation. The model for the financial market is:

$$dB(t) = rB(t)dt \tag{1}$$

$$dS(t) = \mu S_t \; dt + \sigma S_t \, dW_t \tag{2}$$

[3] [7]

B is the price process of a risk-free asset; S is the process of a stock, μ is some interest component times the common price of the stock, and t is the time. The constants $r, \mu \; and \; \sigma$ are deterministic given.

Where: r : is the risk-free interest rate, σ : is the volatility and W refers to Wiener process, it is a real-valued continuous-time stochastic process named in honor of the American Mathematician Norbert Wiener, it is also known as a standard Brownian motion. The wiener process W_t is characterised by the following properties:

1. $W_t = 0$
2. W has independent increments: for every $t > 0$, the future increments $W_{t+u} - W_t$, $u \geq 0$, are independent of the past values W_s ,$0 \leq s \leq t$.

3. W has Gaussian increments: $W_{t+u} - W_t$ is normally distributed with mean 0 and variance u, $W_{t+u} - W_t \sim N(0,u)$.
4. W has continuous paths: W_t is continuous in t.

The Black-Scholes model makes certain assumptions:

- No dividends are paid out during the life of the option
- Markets are random, that means that market movements cannot be predicted.
- The risk-free-rate and volatility of the underlying asset are known and constant.
- The returns of the underlying asset are normally distributed.
- The Black-Scholes model is only used to price European options, which can only be exercised at expiration.
- No arbitrage opportunity, which means that there is no way to make a riskless profit for the trader. [7] [1] [3]

The Black-Scholes equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (3)$$

Let $V(S,t)$ be the price of the option as a function of the underlying asset S at time t. There is no stochastic term in this equation, which gives us a partial differential equation, rather than a stochastic differential equation. V is a "sufficiently regular" function (i.e., continuously differentiable with respect to t and twice continuously differentiable with respect to S). Solving the Black-Scholes differential equation can be done by using the right boundary conditions. To price a European call option, the boundary conditions should be specified (the behaviour of the solution while the values of the asset are changing during time, usually at $S = 0$ and $S \to \infty$, and initial $t = 0$, or final conditions $t = T$ when the option expires).

Also:

$$V = max(S_t - K, 0) \qquad (4)$$

Where K is the strike price.

In the domain:

$$[0, S_{max}] \times [0, T] \tag{5}$$

In particular:

Formula for call options:

$$C(S,\tau) = S_0 N(d_1) - Ke^{-r\tau} N(d_2) \tag{6}$$

C (0, τ) =0 for all τ

C (S, τ) → S – K as S →∞

C (S, τ) = max {S – K, 0}

[3]

Where:

$$d_1 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{T}} \tag{7}$$

$$d_2 = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{\tau} \tag{8}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy \tag{9}$$

S_0: underlying stock price (current stock)

K: strike price of the option also known as the exercise price

σ: volatility

r: risk free interest rate

$\tau = T - t$, it is the time until expiration of the option

T: is the time of option expiration

N: a normal distribution.

This figure below shows the price of a European call option $C(S_t) = \max(0, S_t - K)$, using the strike price K= 40 and $0 \leq S \leq 200$.

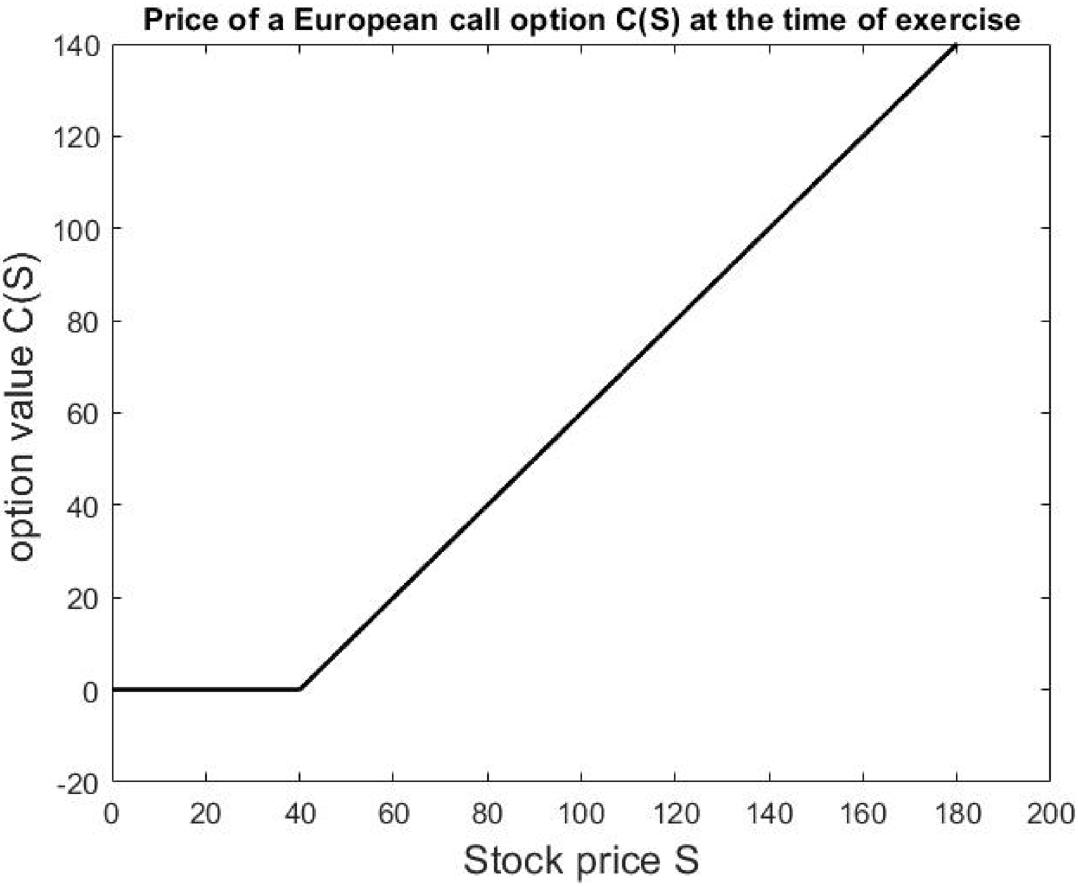


Figure 1 Price of a European Call Option C(S) at the time of exercise

The stock should worth more than the strike price (40 USD) in this example, on the day of exercise, so that the option could be exercised. The profit is the difference between the stock price and the exercise price. But in the case where the stock worth less than the strike price, exercising the option would not have any meaning. Also $S \geq 0$, comes from the fact that the stock prices follow a lognormal distribution because the value of the asset cannot be negative (they are bounded by zero). The stock price S can go to infinity. [8] [3] [9]

2.1.1 The analytical solution of the Black-Scholes equation

We mention two cases:

1. There exists an analytic solution when:
   - Black-Scholes is one-dimensional, (the option depends only on one stock).
   - The volatility and the interest rate are constant.
2. No analytic solution exists if:
1. Black-Scholes equation is multi-dimensional, (the option depends on more than one stock)

The analytical solution for the Black-Scholes in one-dimensional is given by the equation (6)

2.1.2 Black-Scholes equation in higher dimensions

As we said in the second case, when the option depends on more than a stock, the Black-Scholes equation is multi-dimensional, for example, if it depends on two different stocks, we will have a 2-dimensional Black-Scholes equation, hence the dimension of Black-Scholes equation is related to the number of stocks on which an option depends. Black-Scholes 2-dimensional equation is [7] [10]

$$\frac{\partial V}{\partial t} + s_1 r \frac{\partial V}{\partial s_1} + s_2 r \frac{\partial V}{\partial s_2} + \frac{1}{2} [\sigma\sigma^*]_{11} s_1^2 \frac{\partial^2 V}{\partial s_1^2} + \frac{1}{2} [\sigma\sigma^*]_{22} s_2^2 \frac{\partial^2 V}{\partial s_2^2} + [\sigma\sigma^*]_{12} s_1 s_2 \frac{\partial^2 V}{\partial s_1 \partial s_2} - rV = 0 \qquad (10)$$

Black-Scholes d space-dimensional equation is

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{d} r s_i \frac{\partial V}{\partial s_i} + \frac{1}{2} \sum_{i,j=1}^{d} [\sigma\sigma^*]_{ij} s_i s_j \frac{\partial^2 V}{\partial s_i \partial s_j} - rV = 0 \qquad (11)$$

Where

S_i, S_j are the stocks and $i \neq j$

σ^* is the transpose of the symmetric matrix σ . The matrix in 2-dimensions is

$$\sigma = \begin{matrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{matrix}$$

σ_{11} refers to the volatility for stock 1 and σ_{22} the volatility for stock 2. σ_{12} *is the correlation between the two stocks. With the condition*

$$V(s_1, s_2, T) = max\left(\frac{s_1 + s_2}{2} - K\ , 0\right) \tag{12}$$

In Higher dimensions where $d > 1$, There is no general analytic solutions for the equation.

2.1.3 Solving a European Call Option

To price a European Call Option $C(S,t)$ we start by solving the heat equation since the Black-Scholes EDP can be reduced to this equation. Where S represents the stock price, and $t \in [0, T]$, T is the expiry date. We have the following equation:

$$\frac{\partial C}{\partial t}(S,t) + rS\frac{\partial C}{\partial S}(S,t) + \frac{\partial^2}{2}S^2\frac{\partial^2 C}{\partial S^2}(S,t) = 0, \forall\ t \in [0,T], S \in R_+\ , \tag{13}$$

$$C(S,t) = (S-K)^+ = max(S-E,0)\ , \forall\ S \in\ R_+$$

To solve this equation, we have to make various changes of variable to reduce equation (13) to a heat equation of the type:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{14}$$

$$u(x,0) = u_0(x)$$

To be able to do that, we start by deleting the coefficients $S\ and\ S^2$ of the Black-Scholes equation.

Let: $S = Ke^x$, $t = T - \frac{2\tau}{\sigma^2}$, $C(S,t) = Kv(x,\tau)$.

Here we get the condition at $t = 0$ and not at $t = \tau$.

So, we have:

$$v(x,\tau) = \frac{1}{E}\, C\left(Ke^{x}, T - \frac{2\tau}{\sigma^2}\right) = \frac{1}{K}\, C(S,t) \tag{15}$$

We derive (15) with respect to x:

$$\frac{\partial v}{\partial x} = \frac{1}{K}\, \frac{\partial C}{\partial S}\frac{\partial S}{\partial x} + \frac{1}{K}\, \frac{\partial C}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{S}{E}\frac{\partial C}{\partial S}$$

We derive again with respect to x:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{K} S\, \frac{\partial C}{\partial S}\right)$$

$$= \frac{\partial}{\partial S}\left(\frac{1}{K} S\, \frac{\partial C}{\partial S}\right)\frac{\partial S}{\partial x}$$

$$= \frac{1}{K}\left(\frac{\partial C}{\partial S} + S\frac{\partial^2 C}{\partial S^2}\right) S$$

$$= \frac{S}{K}\, \frac{\partial C}{\partial S} + \frac{S^2}{E}\, \frac{\partial^2 C}{\partial S^2}$$

Now we derive (15) with respect to τ :

$$\frac{\partial v}{\partial \tau} = \frac{1}{K}\, \frac{\partial C}{\partial t}\, \frac{\partial t}{\partial \tau} = \frac{1}{K}\, \frac{\partial C}{\partial t}\, \frac{-1}{\frac{1}{2}\sigma^2}$$

For more clarity we introduce the following values:

$$C_t = -\frac{K}{2}\sigma^2 v r \tag{16}$$

$$SC_s = Kv_x \tag{17}$$

$$S^{2C_{ss}} = Kv_{xx} - SC_s = Kv_{xx} - Kv_x \tag{18}$$

Let’s recall the Black-Scholes equation:

$$C_t + \frac{1}{2}\, \sigma^2 S^{2C_{ss}} + rSC_s - rC = 0 \tag{19}$$

We inject (16),(17)and (18) into (19) , Which gives us:

$$-\frac{K}{2}\,\sigma^{2v_\tau}+\frac{K}{2}\,\sigma^2(v_{xx}-v_x)+rKv_x-rKv=0$$

By dividing by $\frac{E}{2}\sigma^2$:

$-v_\tau+v_{xx}-v_x+Fv_x-Fv=0 \quad avec\ F=\frac{2r}{\sigma^2},$

Or using the initial notations:

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(F-1)\frac{\partial v}{\partial x}-Fv \quad avec \quad F=\frac{2r}{\sigma^2} \qquad (20)$$

We then have as initial condition ($\tau=0$, since the condition is at $t=T$:

$$v(x,0)=\frac{1}{K}C(Ke^x,T)=\frac{1}{K}\max(Ke^x-K,0)=\max(e^x-1,0).$$

To get an equation like the heat equation, we have to proceed to a second change of variables, let:

$$v(x,\tau)=e^{\alpha x+\beta\tau}\,u(x,\tau)$$

We reinject all this in the equation (20) :

$$e^{\alpha x+\beta\tau}\left(\beta u+\frac{\partial u}{\partial \tau}\right)=e^{\alpha x+\beta\tau}\left(\alpha^2u+2\alpha\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+(F-1)\left(\alpha u+\frac{\partial u}{\partial x}\right)-Fu\right).$$

$$\beta u+\frac{\partial u}{\partial \tau}=\alpha^2u+2\alpha\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+(F-1)\left(\alpha u+\frac{\partial u}{\partial x}\right)-Fu.$$

By regrouping the terms of the same derivatives:

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}+(\alpha^2+(F-1)\alpha-F-\beta)u+(2\alpha+F-1)\frac{\partial u}{\partial x}.$$

To reduce to the case of the heat equation, we must eliminate the terms in $u\ and\ \frac{\partial u}{\partial x}$, by solving the following system:

$$\begin{Bmatrix}\beta=\alpha^2+(F-1)\alpha-F,\\ 2\alpha+F-1=0.\end{Bmatrix}$$

$$\Leftrightarrow$$

$$\begin{Bmatrix} \alpha = -\frac{1}{2}(F-1) \\ \beta = -\frac{1}{4}(F+1)^2 \end{Bmatrix}$$

We then have:

$$v = e^{-\frac{1}{2}(F-1)x-\frac{1}{4}(F+1)^2\tau}u(x,\tau).$$

Where u verify:

$$\begin{Bmatrix} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \forall x \in R, \forall \tau > 0 \\ u(x,0) = u_0(x) = e^{\frac{1}{2}(F-1)x}\max(e^x - 1,0) = \max\left(e^{\frac{1}{2}(F+1)x} - e^{\frac{1}{2}(F-1)x}, 0\right). \end{Bmatrix}$$

The solution is:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty} u_0(s)e^{-\frac{(x-s)^2}{4\tau}}ds. \tag{21}$$

To evaluate the Option price, we will be using the equations (6) (7)(8).

2.2 Volatility

Volatility estimation is essential in financial mathematics to calculate the price of an option. It measures the volatility of the price of a financial asset. The greater the volatility, the more unstable the asset, if the volatility is zero, we can know exactly the value of the asset in the future. [1]

There are two methods of calculating volatility.

2.2.1 Implicit volatility

Using observed prices C_t of options and inverting the formula of Black-Scholes, we can find the parameter σ. Here, in general, we do not have a single value of sigma, but a curve which depends on the strike K, it is the phenomenon of the "Volatility smile".

2.2.2 Historical volatility

Using historical data of the underlying asset S_t and statistical estimation methods for the mean and the variance, we deduce the parameter of volatility.

2.2.3 Calculation of implicit volatility

The formula of implicit volatility is obtained according to the black-Scholes formula for a European call

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

And $$\sigma = \sqrt{\frac{ln\left(\frac{K}{S}\right) - r(T-t)}{(T-t)(K-\frac{1}{2})}}$$

We therefore have an expression of the volatility as un function of the other parameters. By knowing the previous values of the call, we deduce the volatility. We note that the implied volatility depends on the strike and the date of expiry, we therefore obtain a curve that has the shape of a smile, hence the “volatility smile” phenomenon.

In practice, to calculate the implied volatility, we do not use this formula which is too complicated to set up, we use a numerical resolution. If we call C_0 the value of a call at the initial time, to calculate the implied volatility, we have to solve:

$$C_0 - SN(d_1) - Ke^{-rT}N(d_2) = 0$$

2.2.4 Calculation of historical volatility

There are several formulas to calculate historical volatility. One of them consists of a standard deviation calculation based on the past values of the Call.

Let C_i denote the value of a European Call at the end of the i^{th} period, C the average of these values between time 1 and n, we have:

$$\sigma_n = \sqrt{\frac{\sum_{i=1}^{n}(C_i - \bar{C})}{n}} \qquad , with\ \bar{C} = \frac{\sum_{i=1}^{n} C_i}{n}$$

2.2.5 Plot of the obtained explicit solution

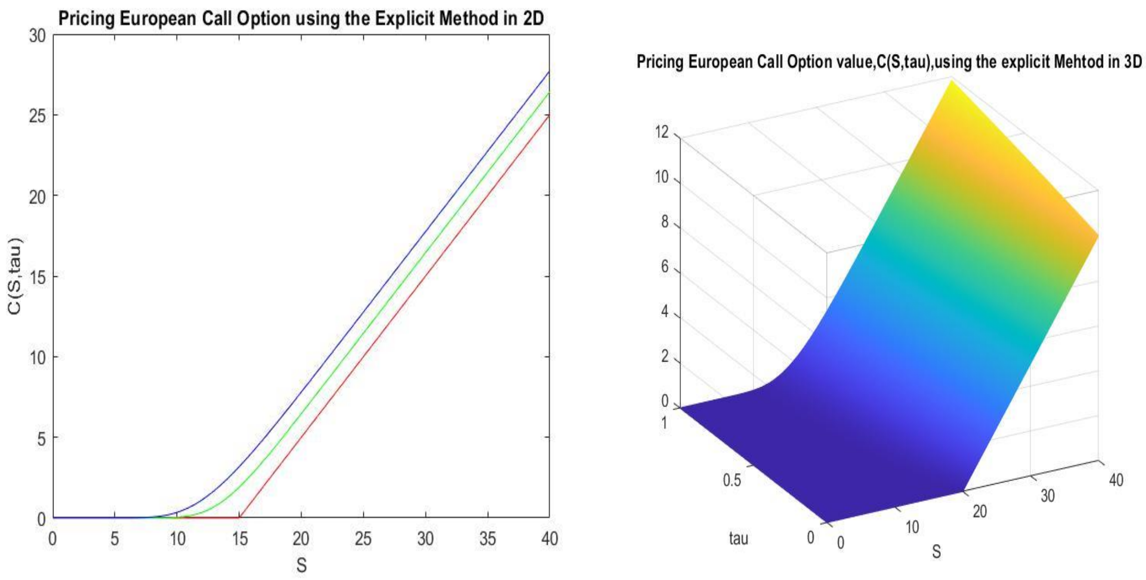


Figure 2Pricing European Call using the explicit Method (in 2D and 3D)

The blue curves represent the value of the European Call Option at expiry date, the green ones half a year before, and the red ones one year before (when the contract is signed).

[11]

3 Numerical Techniques for solving the Black-Scholes equation

Numerical Techniques are the design and study of algorithms to obtain solutions to sets of equations from models from physics, biology, finance.... The models considered are composed of a set of equations for which we do not know how to determine explicit solutions. Therefore, we use some numerical methods to obtain an approximate solution, calculated using the computer.

Solving the Black-Scholes model of A European call option pricing, in higher dimensions, when we have more than a stock, requires one of those methods, to have an approximated solution.

For a European Call with strike K, and expiry date T, we have the following Black-Scholes equation, in which we made the change of variable $\tau = T - t$ to reduce to a condition at $\tau = 0$.

$$\begin{cases} -\frac{\partial C}{\partial t}(S,t) + rx\frac{\partial C}{\partial S}(S,t) + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}(S,t) - rC(S,t) = 0\,, \forall t \in [0,T]\ \ , S \in [0,L], \\ C(S,0) = (S-K)^+ = max(S-K,0)\,, \forall S \in [0,R] \end{cases} \qquad (22)$$

3.1 FINITE DIFFERENCES

In numerical analysis, an important application of finite differences is the numerical resolution of differential equations and partial differential equations: the idea is to replace the derivatives appearing in the equation by finite differences which approximate them. The various methods that result are called finite difference methods. Here we are going to use the Backward Euler method (or Implicit method) to solve our PDE. The Implicit method consists in seeking the approximate value at time t_{n+1} with the following relation:

$$Y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

Therefore, this method consists in taking the derivative at the end of the interval $[t_n, t_{n+1}]$, instead of taking it at the beginning. The value y_{n+1} is obtained by solving an equation. In the case of a differential system, the number of equations to be solved is equal to the numbers of unknown functions. [12] [13] [14]

3.2 Chebyshev Pseudo-Spectral Method

The Chebyshev Pseudo-Spectral method have an important role in the numerical solution of differential equations. PS methods offers superior results for the solution of partial differential equations. [15]

4 Numerical experiments

4.1 Backward Euler method

We use Backward Euler method, let: $C_i^m = C(S_i, t_m) \quad with$:

$$xj = jh \quad with\ j = 0, \dots, M \ \ and\ h = \frac{R}{M} \ \ space\ step\ ,$$

$$t_n = nk\ with\ n = 0, \dots, N \ \ and\ k = \frac{T}{N}\ time\ step,$$

We obtain:

$$-\frac{C_j^{n+1} - C_j^n}{k} + rxj\frac{C_{j+1}^{n+1} - C_{j-1}^{n+1}}{2h} + \frac{\sigma^2}{2}x^2 j\frac{C_{j+1}^{n+1} - 2C_j^{n+1} + C_{j-1}^{n+1}}{h^2} - rC_j^{n+1} = 0.$$

We also need a boundary condition for x=0 and x=R.

For $x = 0$, we take $C(0,t) = 0 \doteq u\alpha\ , and\ for\ x = L\ , we\ take\ C(R,t) = BS(R, E, T - t, r; sigma) \doteq ub(t)$,where BS is the function that permits to obtain the price of the call .

Let $C_T = (C_1^{n+1}, \dots, C_M^{n+1})^t$, $C_0 = (C_1^n, \dots, C_M^n)^t$, we obtain the following matrix:

$$AC_T = C_0 + B_n$$

That gives the components of C_T in function of those of C_0

Where

$$A = \begin{pmatrix} 1 + k\left(\frac{\sigma x_1}{h}\right)^2 + rk & -\frac{rkx_1}{2h} - \frac{k}{2}\left(\frac{\sigma x_1}{h}\right)^2 & 0 & \dots & 0 \\ \frac{rkx_1}{2h} - \frac{k}{2}\left(\frac{\sigma x_1}{h}\right)^2 & 1 + k\left(\frac{\sigma x_2}{h}\right)^2 + rk & -\frac{rkx_2}{2h} - \frac{k}{2}\left(\frac{\sigma x_1}{h}\right)^2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & -\frac{rkx_M}{2h} - \frac{k}{2}\left(\frac{\sigma x_{M-1}}{h}\right)^2 \\ 0 & \dots & 0 & \frac{rkx_M}{2h} - \frac{k}{2}\left(\frac{\sigma x_M}{h}\right)^2 & 1 + k\left(\frac{\sigma x_M}{h}\right)^2 + rk \end{pmatrix}$$

[12]

And B_n is a null vector, B_m is not null and we have:

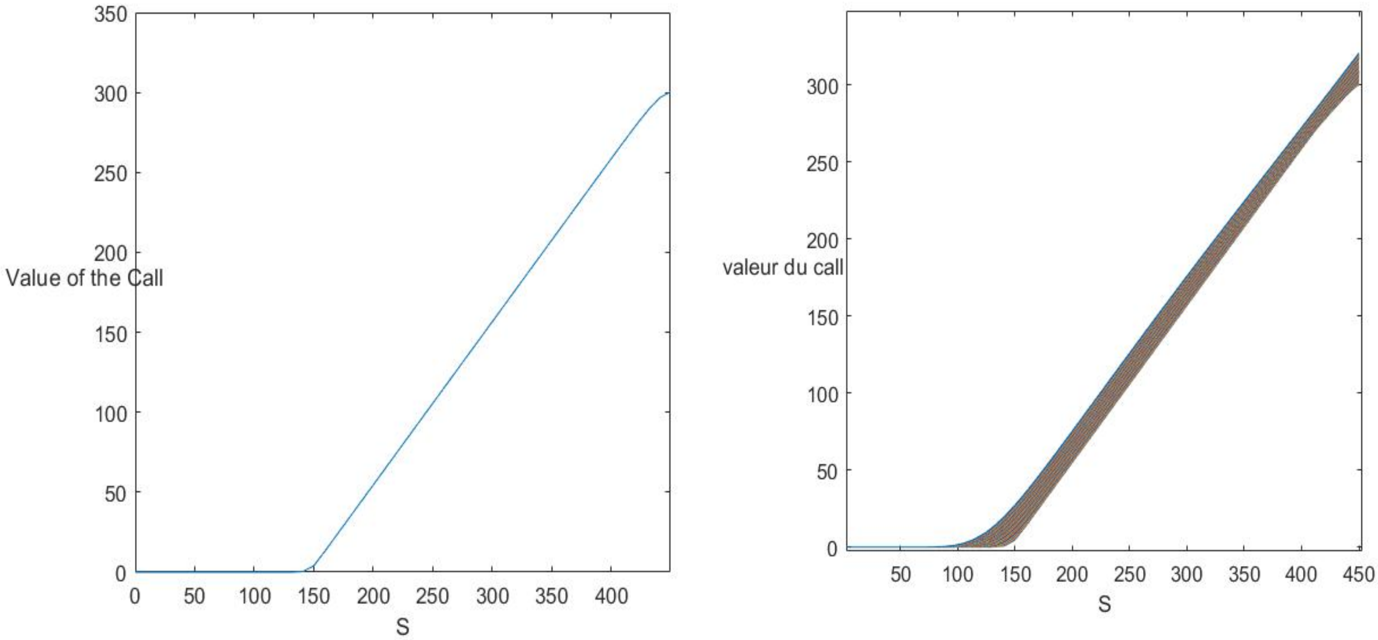
$$B(M) = \left(-\frac{rkx_M}{2h} - \frac{k}{2}\left(\frac{\sigma x_M}{h}\right)^2\right)\ ub\ (nk)$$

Solving the scheme at each time step requires to calculate the solution of the linear system

$$AC_T = C_0 + B_n$$

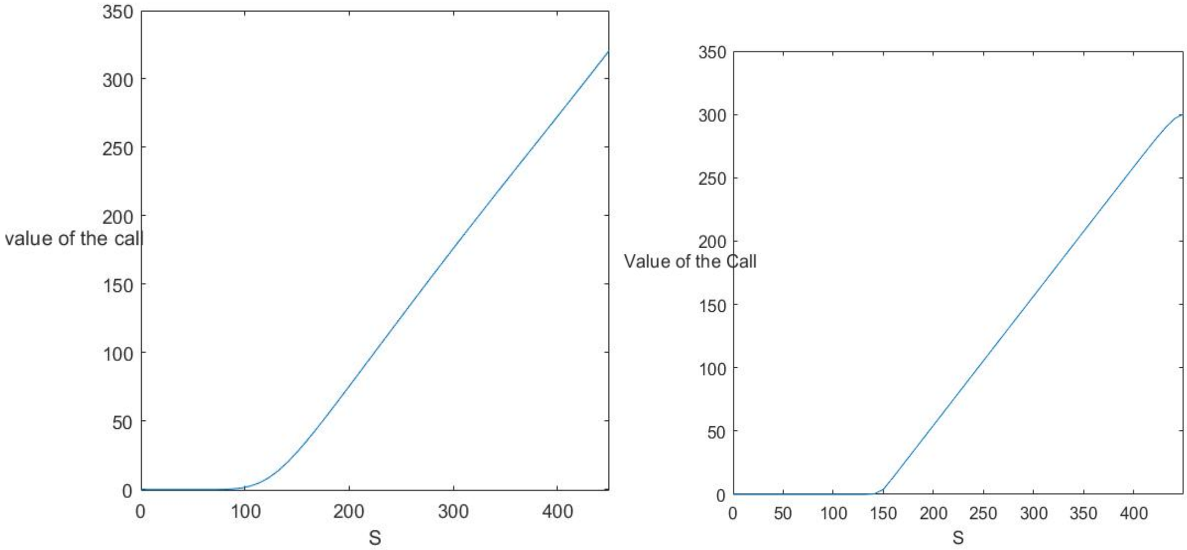
The approximation error of the time derivative being of order 1, and that of the second derivative in space of order 2, it generally follows that the implicit scheme of Euler is in $\vartheta(\delta\tau + \delta x^2)$.

In numerical analysis, the stability of a numerical scheme essentially concerns the numerical behaviour which manifest itself when the steps of temporal and spatial discretization all tend towards 0.



(a) Plot of the solution when t=T (b)Plot of the solution when $t \in]0;T[$

Figure 3 Graphical representation of the values of a European Call Option



(a) Euler implicit (b) Explicit solution

Figure 4 Comparison or results for a European Call where t=0

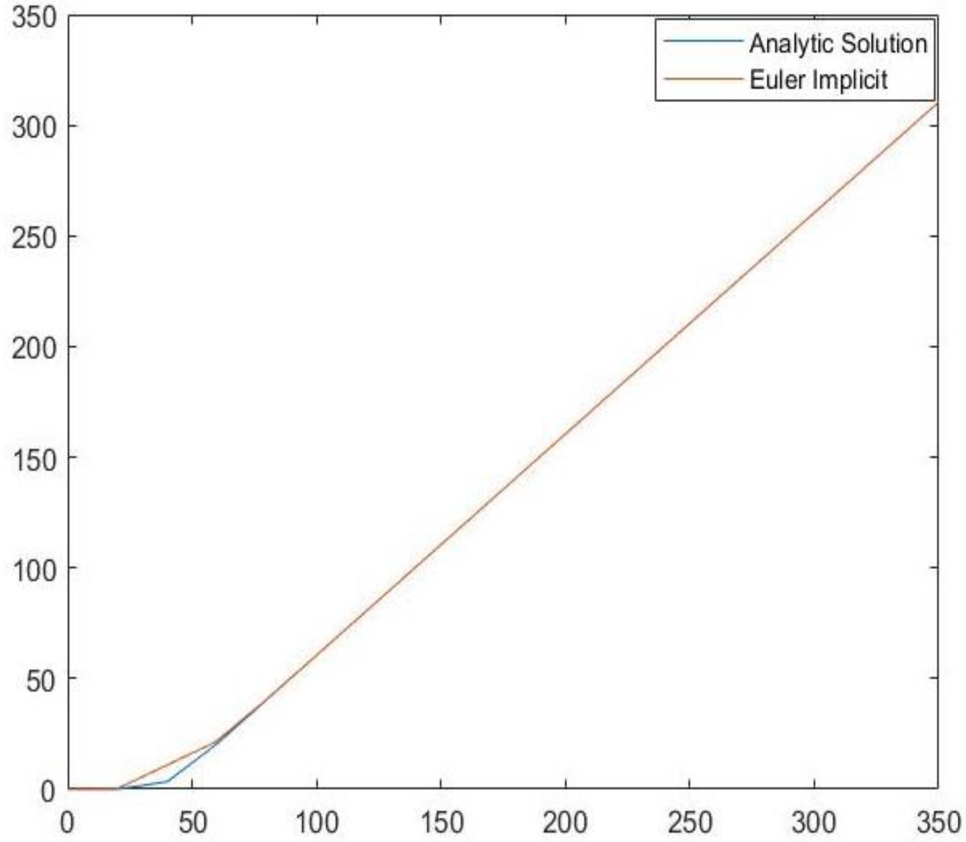


Figure 5 Both Euler Implicit and explicit solution in one figure

The inputs are: $r = 0.0125$, $\sigma = 0.2$, $K = 40$, $T = 1$, $S = 350$

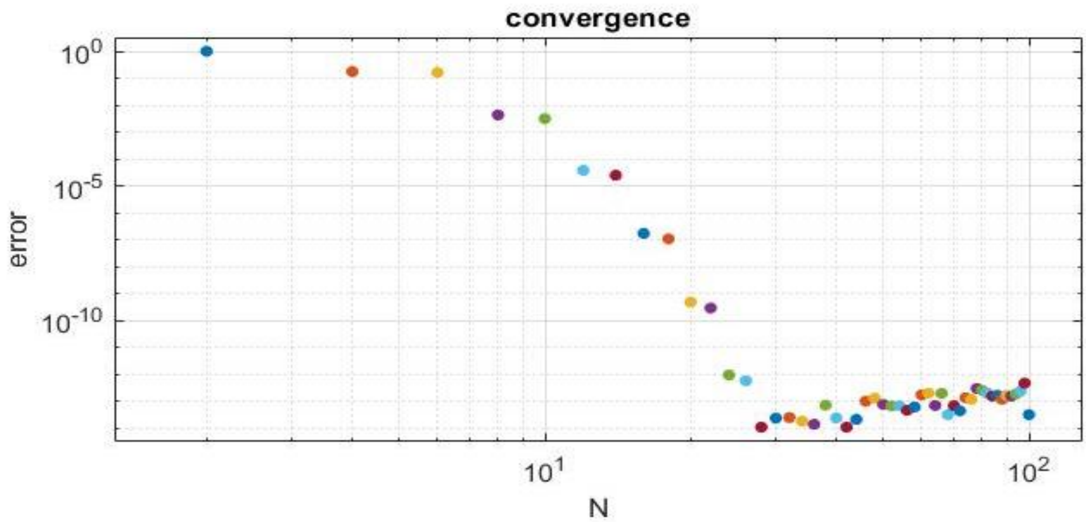


Figure 6 Convergence of Euler Implicit Scheme

Figure 6 shows that the implicit Euler method has a linearly decreasing error as the number of time steps increases. When the number of time steps is multiplied by 10, the error is also multiplied by 10. We thus find theoretical result according to which the method using the implicit Euler in time.

4.2 Spatial discretization using Chebyshev Pseudo-Spectral method

Pseudo-Spectral method to solve partial differential equations requires global basis functions to approach the solution function. The method does not immediately offer high-accuracy quadrature solutions, but it can obtain high order numerical solutions when the latter is smooth. The basic functions used are Chebyshev polynomials, and Chebyshev collocation points for space discretisation.

The rate of convergence of spectral methods for smooth functions is $\vartheta(N^{-m}), with\ m\ \in R$, Where PS methods uses N collocation points, that are usually used to determine an interpolated polynomial of degree $\leq N$. The higher the number of collocation points increases the higher order numerical solutions are obtained by using PS method.

The Black-Scholes equation is a not a smooth function at the initial conditions, since the first derivative is not continuous at K.

We approach the solution by using N equidistant points, that are distributed with the density:

$$density \sim \frac{N}{\pi\sqrt{1-x^2}}. \tag{25}$$

Chebyshev points helps to choose any different types of boundary conditions. They are defined by: [15] [16]

$$x_j = cos\left(\frac{j\pi}{N}\right), \qquad j = 0, \dots\dots, N. \tag{23}$$

The Chebyshev polynomials of the first kind are given by

$$T_n(x) = cos\, n\theta\ ,\ x = cos\,\theta\ , -1 \le x \le 1,\ where\ \ n = 0,1,2, \dots\dots.. \tag{24}$$

The first three polynomials are:

$$\begin{cases} T_0 = 1 \\ T_1 = x \\ T_2 = 2x^2 - 1 \end{cases}$$

They are also orthogonal

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & i \ne j \\ \pi, & i = j = 0 \\ \frac{\pi}{2}, & i = j \ne 0 \end{cases} \tag{26}$$

The orthogonality for discrete function values is given by:

$$\sum_{k=1}^{M} T_i(x_k)T_j(x_k) = \begin{cases} \frac{1}{2} M\delta_{ij}, & i,j \ne 0 \\ M, & i = j = 0 \end{cases} \qquad i,j \le M \tag{27}$$

Since Chebyshev polynomials are defined in $[-1,1]$, and from the domain used in the problem (5) , we obtain:

$$x = \frac{2s}{Smax} - 1 \tag{28}$$

Weierstrass approximation theorem shows that the continuous real-valued functions on a compact interval can be uniformly approximated by Chebyshev polynomials:

$$F(x,t) = \sum_{j=0}^{\infty} c_j(t) T_j(x), \tag{29}$$

Where:

$$c_j =< F, T >= \int_{-1}^{1} \frac{F(x,t)T_j(x)}{\sqrt{1-x^2}}\, dx. \tag{30}$$

L_n is introduced to be able to use(28) with Chebyshev polynomials, It contains the error term, and allows us to obtain:

$$P_n F(x,t) = \sum_{i=0}^{N} \tilde{c}_i(t) T_i(x), \tag{31}$$

$$\tilde{c}_0(t) \equiv \frac{1}{M} \sum_{k=1}^{M} F(x_k, t), \tag{32}$$

$$\tilde{c}_i(t) \equiv \frac{2}{M} \sum_{k=1}^{M} F(x_k, t)\ T_i(x_k), \qquad i = 1, \dots \dots \dots, N \tag{33}$$

$$x_k = cos\left(\frac{\pi\left(k-\frac{1}{2}\right)}{n}\right), \qquad k = 1, \dots \dots \dots, n \quad M \geq N. \tag{34}$$

By inserting the coefficients $\tilde{c}_i$ into equation (31) , and using orthogonality of Chebyshev polynomials we obtain the following expression:

$$P_N F(x,t) = \sum_{i=0}^{N} c_i(t) T_i(x) + \sum_{i=0}^{N} \sum_{k=1}^{M} \sum_{l=N+1}^{\infty} c_l(t) T_l(x_k) T_i(x_k) T_i(x)\,. \tag{35}$$

Therefore, we deduce the coefficients $c_i(0)$, and the initial error will be:

$$\varepsilon = \|F(x_k, 0) - P_N F(x_k, 0)\|$$

$$= \left\| F(x_k, 0) - \sum_{j=0}^{N} c_j(0)\, T_j(x_k) \right\|$$

$$+ \| \sum_{i=0}^{N} \sum_{k=1}^{M} \sum_{l=N+1}^{\infty} c_l(t) T_l(x_k) T_i(x_k) T_i(x) \|, \quad \mathrm{i, k,} = 1, \dots, \mathrm{N}$$

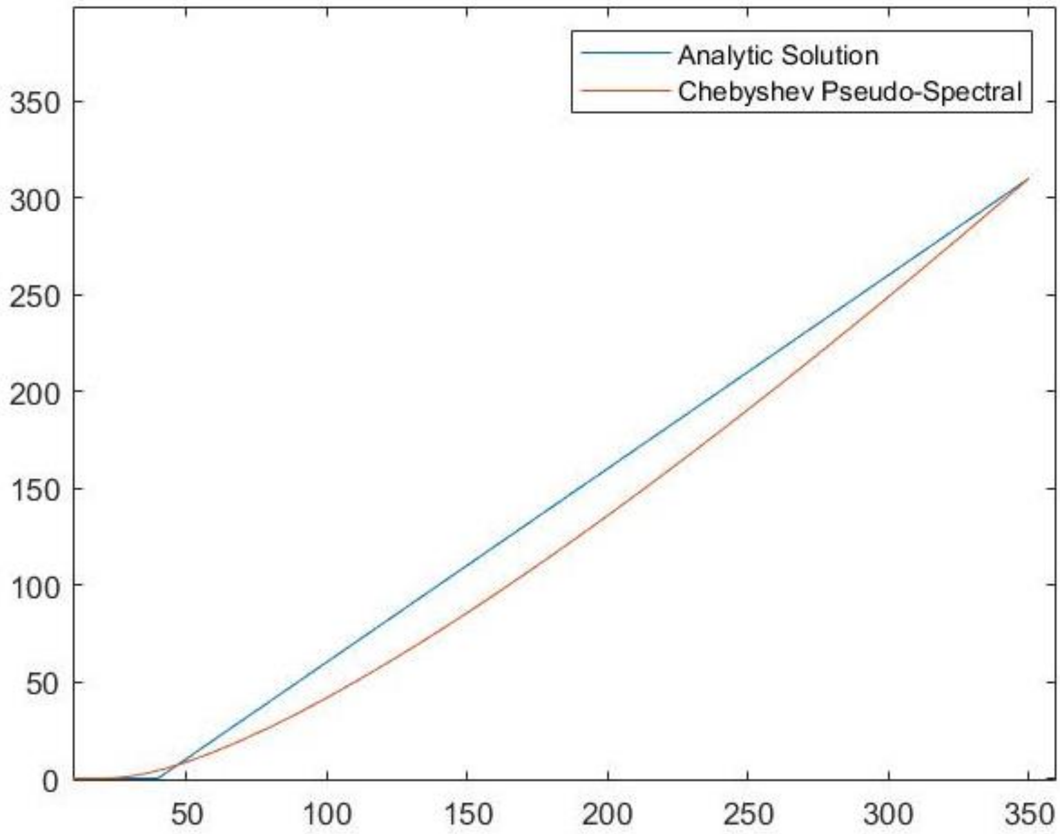


Figure 7 Analytic solution and Chebyshev PS approximation

This figure shows the difference between the exact and the approximate solution, approximated by using Chebyshev Pseudo-Spectral method, this method seems to be more accurate.

4.3 Comparison between Euler Implicit and Chebyshev Pseudo-Spectral method

To compare between the two numerical methods, FD AND PS, we compute the error in Euler Implicit that takes the form of $err_E(N) = \frac{1}{N} = \frac{1}{N^2}$,and Chebyshev Pseudo-Spectral that takes the form of $err_C(N) = e^{-\alpha N}$ and compare between them, we obtain the following figures, after running a MATLAB program.

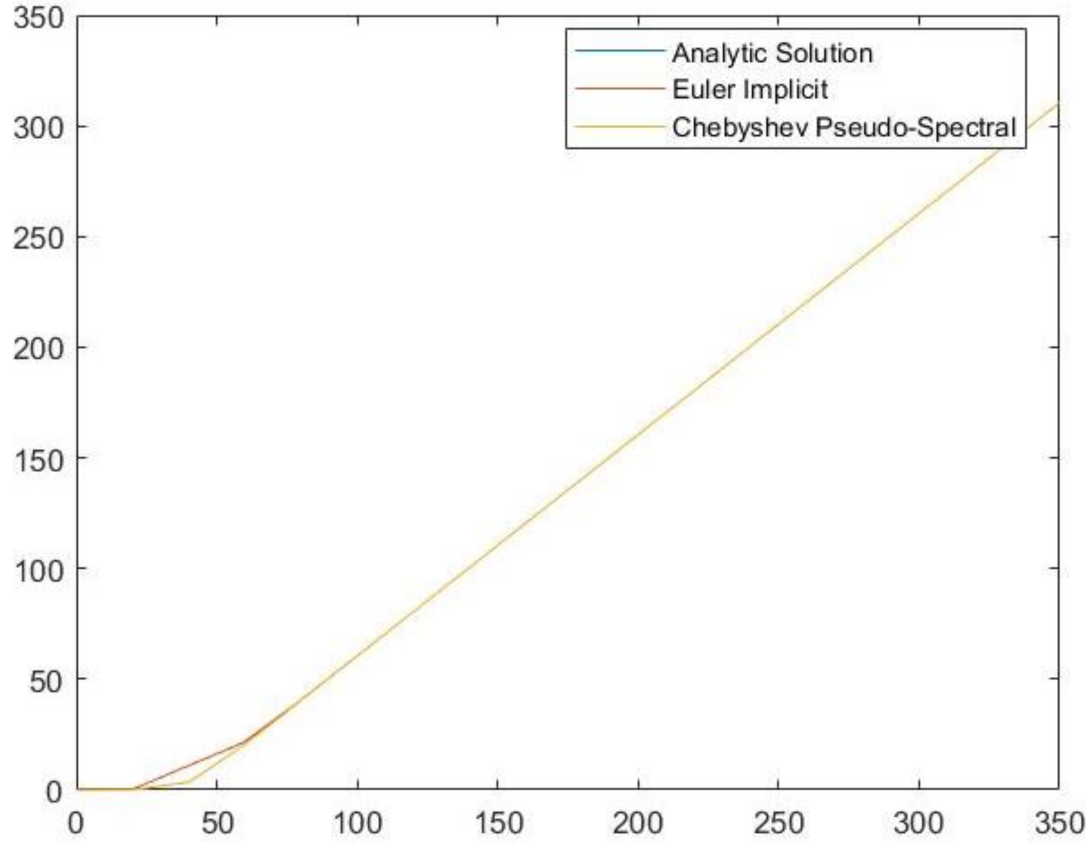


Figure 8 Analytic solution compared to Euler Implicit and Chebyshev PS approximations

This figure shows how Chebyshev Pseudo-Spectral is more accurate than the Euler Implicit, the curve is almost identical with the curve of the analytic solution (the Blue one), we barely can see it, unlike the curve of the Euler Implicit.

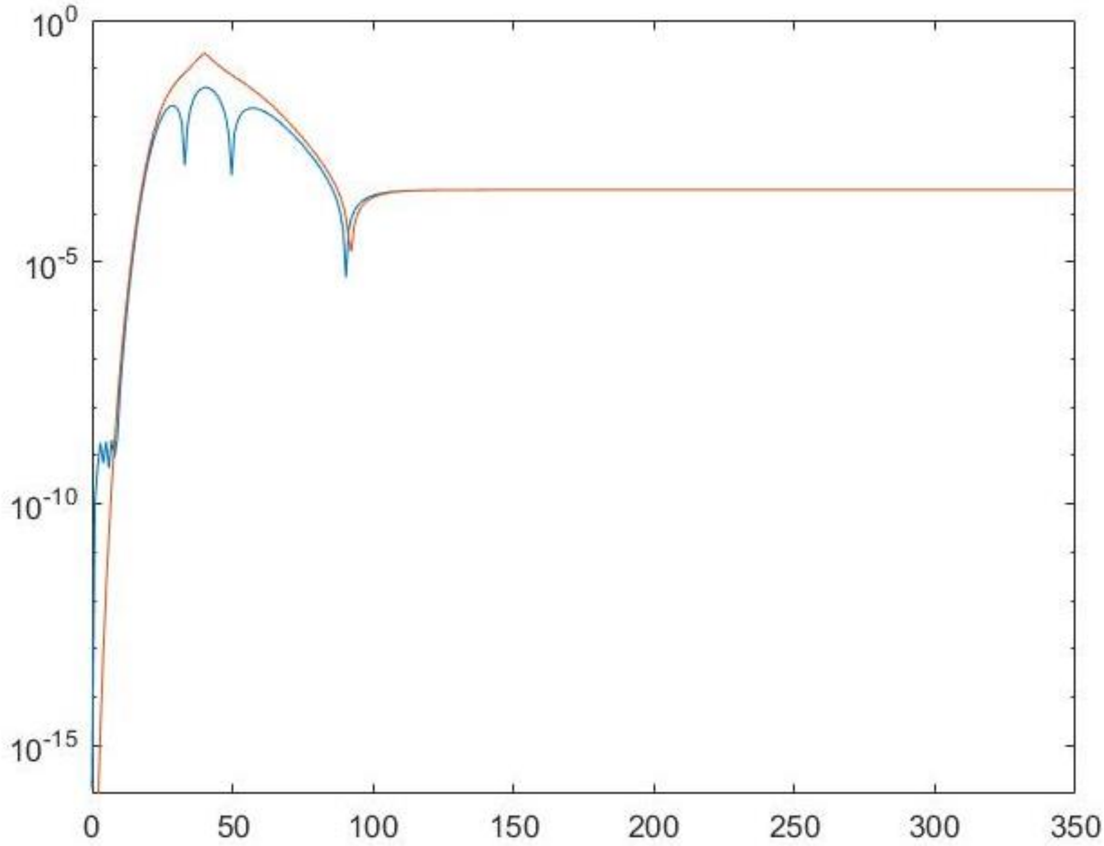


Figure 9 Number of collocation points used here for Euler Implicit N=40, and for Chebyshev PS ,N=20, There's a remarquable difference between both errors .

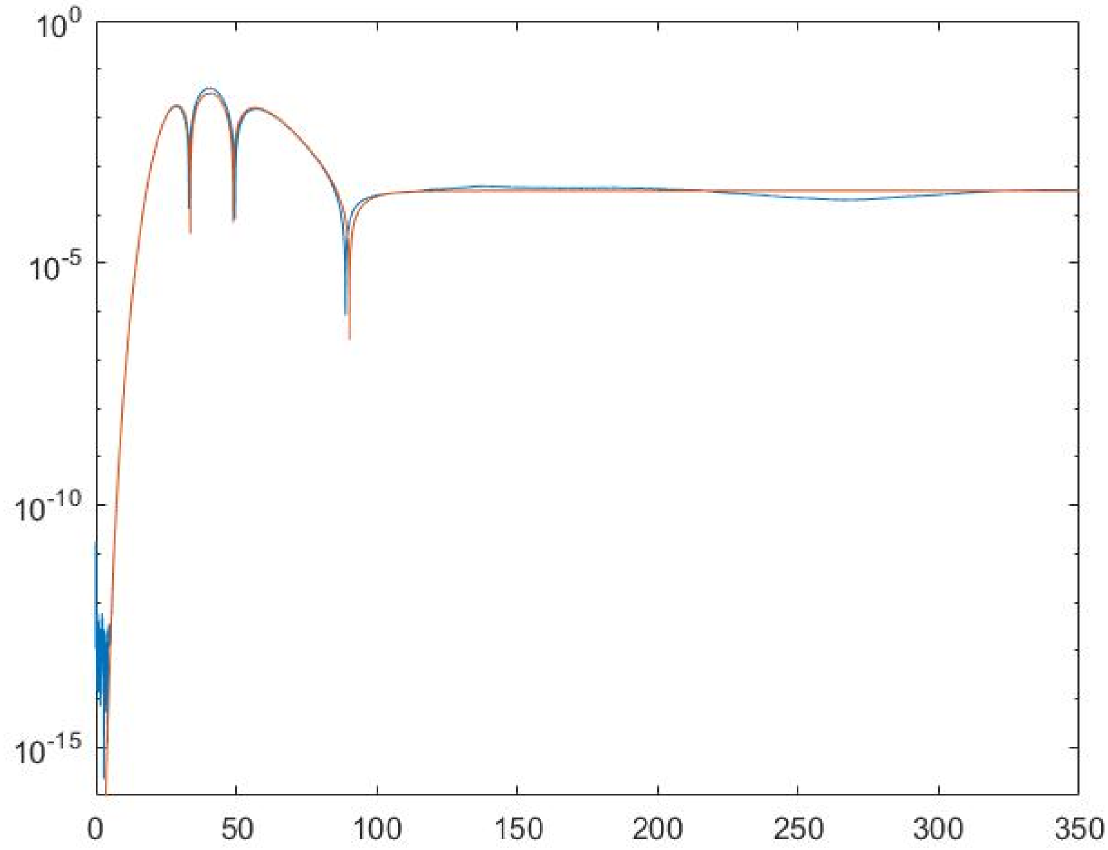


Figure 10 the number of collocation points used are 1000 for Euler Implicit, and 200 for Chebyshev PS

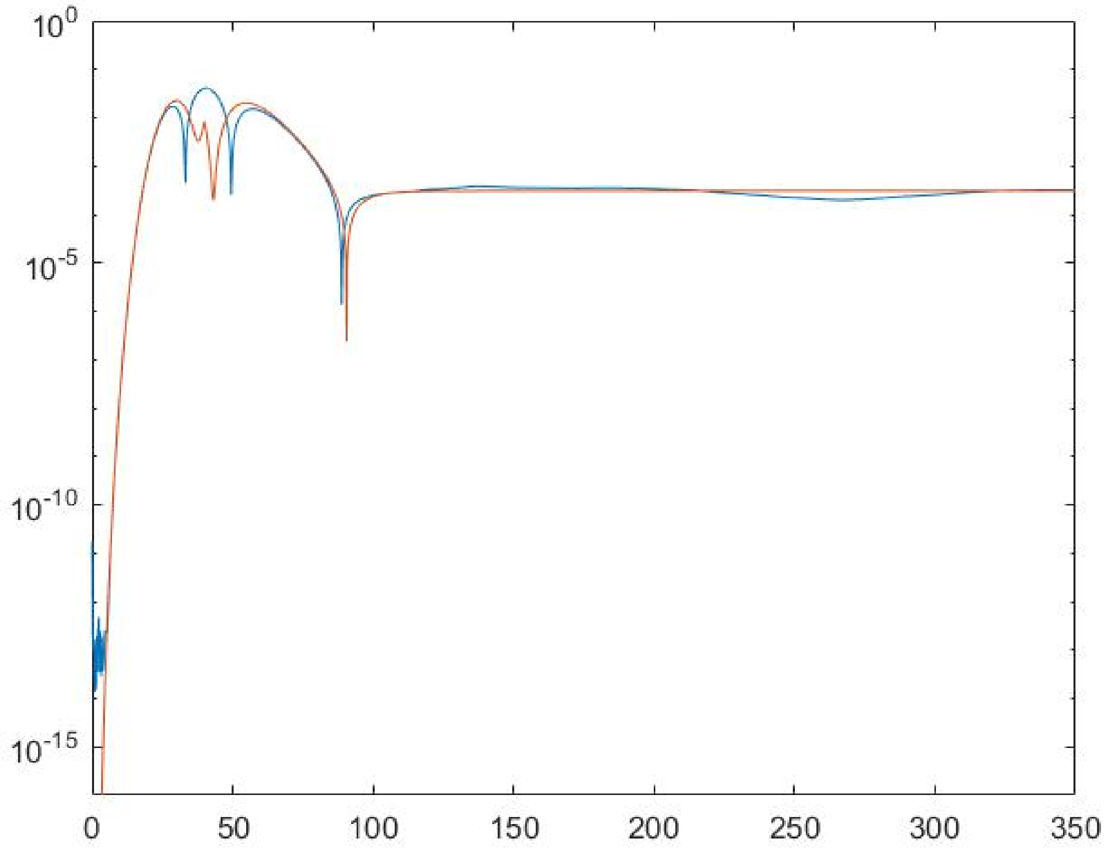


Figure 11 the number of collocation points used is 200 for both methods

4.4 Discussion

This Numerical experiments shows that , in higher dimensions numerical methods need to be used to solve a nonlinear PDE .Solving the Black-Scholes equation , by using the Euler Implicit and Chebyshev Pseudo-Spectral methods , we notice that the PDE solution can be approximated much more accurately by Chebyshev PS than by Euler Implicit, to obtain almost the same error in both method as shown in Figure 10 the number of collocation points used are 1000 for Euler Implicit, and 200 for Chebyshev PS , we had to set N to 1000 in Euler Implicit and only to 200 in Chebyshev PS.

5 Conclusion

The Black-Scholes model is one of the most important model in modern financial theory .It is used for the valuation of stock options ,however , the analytical solution of the Black-Scholes PDE can be obtained only in one dimension, in higher dimensions we need to use numerical methods .Therefore , the purpose of this work was to present the mathematical approach of discretization of the Black-Scholes model for European Call option pricing by reducing it to the heat equation, using both Chebyshev Pseudo-Spectral and Euler Implicit methods , and to implement them. The comparisons carried out by simulation have shown that the number of points in space and the time step have an influence on the accuracy of both methods. The accuracy of the Chebyshev Pseudo-Spectral method approximation is very sensitive to the location in the interface. The study also made it possible to further strengthen our knowledge in this field.

The Chebyshev Pseudo-Spectral method is highly accurate for even small values of N, it is a global and often preferred for high-order accuracy, but also, difficult to program. Unlike Euler Implicit method, which is easier to program, however the method is local and we need bigger values of N to obtain a higher accuracy. [7] [11] [12] [15]

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